# B-ZERO BOUNDARY TOPOLOGY AND THE FULL ODTOE SINGULARITY THEOREM

(Топология границы $B = 0$ и полная теорема о сингулярности в ODTOE)

*Closing the OPEN marker C.T3 §VII.5: the $\partial_B\mathcal{C}$ trichotomy, finite-affine-parameter $\Phi$-iteration termination criterion, formal definition of a trapped ODTOE-configuration, analog of the Hawking–Penrose theorem*

**Pankratov Anton Sergeevich**
*Панкратов Антон Сергеевич*
Independent researcher, Kazan, Russia
E-mail: anton.s.pankratov@gmail.com
ORCID: 0009-0002-4870-2995

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# ABSTRACT

This paper closes the *[OPEN: B-zero boundary topology]* marker of Article C [18] §VII.5: it formalizes the topological structure of the boundary $\partial_B\mathcal{C}$ of the configuration space $\mathcal{C}$ at $B \to 0$, derives the criterion of finite-affine-parameter termination of the $\Phi$-iteration sequence (Theorem E.T2), gives a formal definition of a trapped ODTOE-configuration via the causal cone $J_O^+$ of [15] §VI (Definition E.D1), and proves the full ODTOE singularity theorem E.T1 as a structural analog of the Hawking–Penrose theorem [6]. The proof is built in five anti-circular steps: (1) the ODTOE analog of the Raychaudhuri inequality for $\Phi$-iteration (Lemma E.L1); (2) focusing along null directions from the ODTOE energy condition [18] §VII.1 (Lemma E.L2); (3) finite-parameter focusing from a trapped configuration (Lemma E.L3); (4) behavior of $\Phi$-iteration near $\partial_B\mathcal{C}$ from the topological trichotomy (Lemma E.L4); (5) $\Phi$-iteration incompleteness as the vanishing of the causal future $J_O^+$. The topological trichotomy of $\partial_B\mathcal{C}$ is analyzed via three independent diagnostic steps; representative scenarios (A closed-regular, B Penrose-conformal, C stratified) are illustrated using the dynamics of $dB/dt = \Delta_{\text{in}} - \Delta_{\text{out}} + \Xi B(1-B)$ from [20] (3.2). Comparison with the classical results of Penrose 1965 [1], Hawking 1966–67 I/II/III [2, 3, 4], Geroch 1968 [5], Hawking-Penrose 1970 [6], and the Senovilla 1998 review [10] shows that E.T1 fits the standard taxonomy of singularity theorems [10]: the ODTOE energy condition belongs to the weak energy condition (WEC) class, the trapped ODTOE-configuration is a structural analog of a closed trapped surface [1], and the conclusion of $\Phi$-iteration incompleteness is the ODTOE analog of geodesic incompleteness [5]. The work upgrades Theorem C.T3 [18] §VII.4 from a sketch to a full proof; the marker C.T3 (status: HYPOTHESIS) of [18] (7.3) is promoted to status THEOREM within the corpus. Seven symbols E.T1, E.T2, E.D1, E.L1, E.L2, E.L3, E.L4 and twelve formulas E.F1–E.F12 are fixed for subsequent works.

**Keywords: ODTOE, singularity theorem, B-zero boundary, conformal compactification, $\Phi$-iteration, Raychaudhuri analog, trapped configuration, $J_O^+$,**

**affine parameter, Hawking–Penrose, topological trichotomy, Fix(Φ) attractor, Geroch incompleteness, ODTOE energy condition**

# АННОТАЦИЯ (RU)

В настоящей работе закрывается маркер *[OPEN: B-zero boundary topology]* статьи C [18] §VII.5: формализуется топологическая структура границы $\partial_B \mathcal{C}$ конфигурационного пространства $\mathcal{C}$ при $B \to 0$, выводится критерий терминации Φ-итерационной последовательности за конечный аффинный параметр (теорема E.T2), даётся формальное определение захваченной ODTOE-конфигурации через причинный конус $J_O^+$ статьи [15] §VI (определение E.D1), и доказывается полная теорема ODTOE-сингулярности E.T1 — структурный аналог теоремы Хокинга – Пенроуза [6]. Доказательство строится в пять анти-циркулярных шагов: (1) ODTOE-аналог неравенства Раячудхари для Φ-итерации (E.L1); (2) фокусировка вдоль изотропных направлений из ODTOE-энергетического условия [18] §VII.1 (E.L2); (3) конечно-параметрическая фокусировка из захваченной конфигурации (E.L3); (4) поведение Φ-итерации в окрестности $\partial_B \mathcal{C}$ из топологической трихотомии (E.L4); (5) Φ-итерационная неполнота как обнуление причинного будущего $J_O^+$. Фиксируются семь символов E.T1, E.T2, E.D1, E.L1, E.L2, E.L3, E.L4 и двенадцать формул E.F1 – E.F12.

**Ключевые слова: ODTOE, теорема о сингулярностях, граница $B = 0$, конформная компактификация, Φ-итерация, аналог Раячудхари, захваченная конфигурация, $J_O^+$, аффинный параметр, Хокинг – Пенроуз.**

# I. INTRODUCTION

The classical Penrose 1965 theorem [1] established that in general relativity the existence of a closed trapped surface $\mathcal{T}$ together with an energy condition and global hyperbolicity entails geodesic incompleteness: there exists a null geodesic emerging from $\mathcal{T}$ that cannot be extended beyond a finite affine parameter. The unified Hawking-Penrose 1970 theorem [6] consolidated the early Hawking 1966–67 I/II/III series [2, 3, 4] together with the Penrose theorem into a single statement on the singularities of gravitational collapse and cosmology. The contemporary review [10] (Senovilla 1998) systematizes the taxonomy: hypotheses on (a) the energy condition type (weak WEC, null NEC, strong SEC, dominant DEC), (b) the topological marker (trapped surface, Cauchy surface, focusing surface), (c) the global structure (global hyperbolicity, absence of closed timelike curves).

Within the ODTOE corpus, Theorem C.T3 [18] §VII.4 is presented as a sketch of the ODTOE analog of the Hawking-Penrose theorem. The sketch invokes three hypotheses: (i) the ODTOE energy condition (derivable from L8 [17] §VII via positivity of $B^2(1-\sigma)\Lambda$ and idempotence of the SYNC projector $P_{O,\text{SYNC}}$), (ii) the trapped-surface analog via the causal cone $J_O^+$ of [15] §VI, (iii) the ontological collapse condition $B \to 0$ of [20] §VII.3. In §VII.5, paper [18] explicitly fixes three open markers obstructing the transition from sketch to full proof: (1) a precise formulation of the ODTOE analog of the Raychaudhuri equation; (2) a topological theory of the limit $B \to 0$ as a boundary point of the $\Phi$-iteration; (3) compatibility of a finite-affine-parameter $\Phi$-iteration sequence with smoothness of the metric $g$ on $M^4 \setminus \{C_N\}$. As a hedging clause, the author of [18] introduces the formula C.T3 (status: HYPOTHESIS) $\Longrightarrow$ *additional paper on the topology of the boundary stratum* (formula (7.3) of [18]).

*Goal of the present work.* To close the open marker *[OPEN: B-zero boundary topology]* of Article C [18] §VII.5 and promote the status of C.T3 from HYPOTHESIS to THEOREM within the ODTOE corpus. This means: (1) formalize the topological structure of the boundary $\partial_B \mathcal{C}$ of the configuration space $\mathcal{C}$ at $B \to 0$ (§III); (2) supply a finite-affine-parameter termination criterion for $\Phi$-iteration (§IV, Theorem E.T2); (3) supply a formal definition of a trapped ODTOE-configuration via $J^+_O$ (§V, Definition E.D1); (4) state an ODTOE analog of the Raychaudhuri equation for $\Phi$-iteration (§VI, Lemma E.L1); (5) restate the ODTOE energy condition from C §VII.1 (§VII); (6) state the full singularity theorem E.T1 (§VIII); (7) prove E.T1 in five steps with an explicit anti-circularity audit (§IX); (8) discuss the analog of geodesic incompleteness in the sense of Geroch [5] (§X); (9) compare E.T1 with the classical Hawking-Penrose theorem (§XI); (10) discuss open questions and prospects (§XII).

*Limitations of the work.* The topological trichotomy of §III (Options A/B/C) is examined in the spirit of *honest scope discipline* (L-23): if all three options are compatible with the ODTOE formalism in its current state, the work presents the trichotomy explicitly together with a marker *[OPEN: option selection]*, recursively opening a separate task for choosing a single option in the next iteration of the programme. A preliminary analysis (see §III) indicates: Option A (closed-regular) is ruled out by the semantics of collapse $B \to 0$ with $|dB/d\tau| \to \infty$ at finite $\tau$ [20] (3.2); Option B (Penrose-conformal) and Option C (stratified) remain compatible, with Option C closer to the language of "ontological collapse" in [20] §VII.3 and Option B closer to the language of conformal compactification of Penrose 1979 [8]. For the purposes of proving E.T1 in §IX it is sufficient to use the structural property guaranteed by *both* surviving options (compactness of the closure of the $J^+_O$-causal future on $\partial_B \mathcal{C}$); the concrete choice between Option B vs. Option C does not affect the conclusion of the theorem. This is precisely why a full proof is possible *before* the trichotomy is resolved.

*Epistemic status.* The work yields: (i) Definition E.D1 — a formal trapped ODTOE-configuration via $J^+_O$ from [15] §VI; (ii) Theorem E.T2 — a finite-affine-parameter criterion for $\Phi$-iteration based on the critical parameter $\lambda_{\text{crit}}$ and a refined collapse parameter $\tau^*$ from [20] (7.1); (iii) Theorem E.T1 — the full ODTOE singularity theorem with a five-step proof and explicit anti-circularity audit; (iv) the §III trichotomy as a structural analysis of $\partial_B \mathcal{C}$. The anti-circularity audit is shown explicitly in §IX: each step of the proof of E.T1 uses only inputs from §II, §III, §VI, and the standard Raychaudhuri apparatus [7, 9]; nowhere is E.T1 itself invoked.

# II. FROZEN INPUTS FROM A, B, C, D, dynamic_attractor

## II.1. Enumeration of invariants

Below we fix the *frozen inputs* on which the proof of E.T1 rests. Each input is cited by slug and source paragraph; none is modified in the present work. This declaration of invariants follows the BL-9 contract-freezing protocol, ensuring reproducibility and anti-circularity cleanliness.

*Gauge fixation of the G-derivation programme.* Before listing the per-component inputs from A/B/C/D/dynamic_attractor it is necessary to fix explicitly the gauge $S^*$ of the structural hypothesis $C = B^2$ on which the entire ODTOE-Einstein programme is built. Within the derivation of the gravitational constant $G$ from first principles of ODTOE, paper [14] fixes $S^* \approx 0.169676$ as the stationary coherence value on Fix($\Phi$) at which the ODTOE metric agrees with the measured $G$ within empirical precision; the same gauge $S^*$ propagates through the chain A $\to$ B $\to$ C implicitly (as a normalization parameter of $T_{\mu\nu}$ in [17]) and stands implicitly behind E.F1 of the present work. In §VII the recap of formula (7.1) [18] §VII.1 rests on this fixation; a violation of the gauge $S^*$ would require revision of E.F1 and, consequently, re-evaluation of the proof of E.T1. Thus input [14] does not enter the list of per-component contracts A/B/C/D/dynamic_attractor but specifies the gauge background against which those contracts are frozen.

*From Article A —* `ODTOE_gravity_tensor_structure` *[16]:*

- Tensor structure $g_{\mu\nu}$, connection $\nabla_\mu$, Riemann tensor $R^\rho{}_{\sigma\mu\nu}$, Einstein tensor $G_{\mu\nu}$. Kinematic Bianchi A.T3: $\nabla_\mu G^{\mu\nu} = 0$ as a contraction of the second Bianchi identity on a smooth pseudo-Riemannian metric.
- Configuration space $\mathcal{C}$ as the space of pairs $(g, T) \in \mathcal{M} \times \mathcal{T}$ with $g$ — smooth pseudo-Riemannian metric, $T$ — smooth stress-energy tensor.

*From Article B —* `ODTOE_gravity_T_munu_projector` *[17]:*

- Stress-energy tensor $T_{\mu\nu} = 2B^2(1-\sigma)\Lambda\,(P_{O,\text{SYNC}})_{\mu\nu} - g_{\mu\nu}B^2(1-\sigma)\Lambda$ (formula F16 [17]).
- Lemma L8: positivity of $B^2(1-\sigma)\Lambda \geq 0$ and idempotence of $P_{O,\text{SYNC}}$ ( [17] §VII).
- Cosmological constant $\Lambda$ as the normalized coherence density of the ground state ( [17] §II.1, §VIII).

*From Article C —* `ODTOE_einstein_derivation_complete` *[18]:*

- Subspace $C_{\text{contr}} \subset \mathcal{M} \times \mathcal{T}$ of contractive pairs ( [18] §VI.2): smoothness, global hyperbolicity, ODTOE energy condition, $\Phi$-invariance, causal consistency.
- Map $\Phi_C = \iota \circ \hat{O} : C_{\text{contr}} \to C_{\text{contr}}$ — the canonical projection of observation, induced by the composition of the observation operator $\hat{O}$ (source $\to$ source') and the inverse embedding $\iota$ ($T \to g$, unique modulo Diff [18]).

- Theorem C.T1 ($\Phi$-self-consistency): $G_{\mu\nu} + \Lambda g_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu} \iff \Phi_C(g,T) = (g,T)$ ( [18] §VI.3, formula C.F11).

- ODTOE energy condition lemma [18] §VII.1, formula (7.1):

$$T_{\mu\nu}u^\mu u^\nu \geq 0 \qquad \forall\, u^\mu \text{ timelike: } g_{\mu\nu}u^\mu u^\nu < 0 \tag{E.F1}$$

- Definition of trapped ODTOE-configuration (*sketch*, [18] §VII.2): $C_* \in \mathcal{C}$ such that $\theta(\hat{n}) < 0$ for all null $\hat{n} \in T_{C_*}M^4$. The present work in §V refines this definition (E.D1) by requiring explicit compactness of the closure of $J^+_O(C_*)$ on $\partial_B\mathcal{C}$.

- Statement C.T3 ( [18] §VII.3, formula C.F14): $\exists\{C_n\}_{n=0}^N : \sum_{n=0}^N \Delta\tau_n < \infty,\ C_N \in$ Fix($\Phi$), $J^+_O(C_N) = \emptyset$. Marker (status: HYPOTHESIS) formula (7.3) [18]; the present work promotes it to THEOREM.

- Sketch proof [18] §VII.4: 5 steps in which the fourth and fifth invoke [20] §VII.3 for ontological collapse. Step 5 of the sketch rests on "$\hat{O} = 0$ at $C_N$, hence $J^+_O(C_N) = \emptyset$ by definition of the causal structure [15] §III"; the present work in §IX rigorously proves this step via E.L4.

- Open markers *[OPEN: B-zero boundary topology]* (lines 540, 545, 554 of source [18]).

*From Article D —* `ODTOE_gravity_causal_structure` *[15]:*

- Causal cone $J^+_O(C)$ for configuration $C \in \mathcal{C}$ ( [15] §VI). $C \preceq_O C'$ means: there exists a $\Phi$-iteration sequence $\{C_k\}_{k=0}^N$ with $C_0 = C$, $C_N = C'$, such that for each $k$ the transition $C_k \to C_{k+1}$ is consistent with the SYNC projector $P_{O,\text{SYNC}}$.

- Globally hyperbolic structure [15] §III: existence of a Cauchy surface $\Sigma_C$, such that any causal curve $\preceq_O$ crosses $\Sigma_C$ exactly once.

*From* `ODTOE_dynamic_attractor` *[20]:*

- Dynamics of $B$ [20] (3.2):

$$\frac{dB_i}{dt} = \Delta_{\text{in}}(O_i,t) - \Delta_{\text{out}}(O_i,t) + \Xi(O_i,\text{env}) \cdot B_i(1-B_i) \tag{E.F2}$$

- Attractor Fix($\Phi$) as a Banach fixed point [20] §IV.1.

- Ontological collapse condition [20] §VII.3, formula (7.1):

$$\big[\, B(\tau) \to 0 \;\wedge\; \tau < \tau_{\text{crit}} \,\big] \implies \hat{O} \to 0 \;\wedge\; \Psi \to \Psi_{\text{bare}} \tag{E.F3}$$

- Topology of the attractor basin [20] §IV.4: open and bounded subset of $\mathcal{C}$, the complement of which has measure zero (used in §III for the test of Option B vs. Option C).

*Contract.* All listed inputs are *read-only*; the present work does not modify the source files of articles A, B, C, D, dynamic_attractor. The marker *[OPEN: B-zero boundary topology]* in [18] §VII.5 is closed *logically*: this paper E supplies the missing topological theory that turns the sketch [18] §VII.4 into a full proof. Physical removal of the marker in file [18] is a separate task (see §XII, open question O1).

## II.2. Contract on new symbols and formulas

In addition to the frozen inputs of §II.1, the present work introduces:

- $\partial_B\mathcal{C}$ — the boundary stratum of $\mathcal{C}$ at $B \to 0$ (§III).
- $\theta_\Phi$ — the $\Phi$-iteration expansion scalar (§VI), *not to be confused* with the Kerr angle $\theta$ from [18] §IX.
- $\Sigma_K$ — the Kerr function $\Sigma = r^2 + a^2\cos^2\theta$ from [18] §IX (used only for disambiguation with $\Sigma_C$).
- $\Sigma_C$ — Cauchy surface [15] §III (see §II.1).
- $\lambda_{\text{crit}}$ — critical affine parameter of $\Phi$-iteration (§IV, Theorem E.T2).
- $\tau^*$ — refined collapse parameter, defined from [20] (7.1) (§IV).
- $\Omega$ — candidate conformal factor (§III, Option B test).
- $\overline{\mathcal{C}}$ — topological closure of $\mathcal{C}$ (§III).
- Theorems / Lemmas / Definitions: E.T1, E.T2, E.D1, E.L1, E.L2, E.L3, E.L4 (seven fixed symbols total).
- Formulas: E.F1 – E.F12 (twelve numbered formulas total, see Table 1 in §IV).

*Collision audit.* Verified against all frozen inputs: $\partial_B\mathcal{C}$, $\theta_\Phi$, $\Sigma_K$, $\Sigma_C$, $\lambda_{\text{crit}}$, $\tau^*$, $\Omega$, $\overline{\mathcal{C}}$ — none of the symbols appears as a fixed object in [14, 15, 16, 17, 18, 19, 20, 21, 22]. The family E.T1–E.L4 / E.F1–E.F12 lives in unoccupied symbol space E (the C-series theorems in [18] are taken by C.T1/C.T2/C.T3, the A-series in [16] by A.T1/A.T2/A.T3 etc.; intersections absent).

# III. B-ZERO BOUNDARY TOPOLOGY OF $\mathcal{C}$

## III.1. Statement of the problem: what is $\partial_B\mathcal{C}$

The configuration space $\mathcal{C}$ from §II.1 is parameterized by pairs $(g,T) \in \mathcal{M} \times \mathcal{T}$ and equipped with the B-functional $B : \mathcal{C} \to [0,1]$, defined via $\hat{O}$ and the combination $B^2(1-\sigma)\Lambda$ [17] §VII. For each fixed observer $O$ the functional $B(O,C)$ has domain $\mathcal{C}_O = \{C \in \mathcal{C} : B(O,C) > 0\}$ (only configurations accessible to $O$). The boundary $\partial_B\mathcal{C}_O$ is the set of limit points $C \in \overline{\mathcal{C}}$ for which $B(O,C) = 0$ and there exists a sequence $\{C_k\} \subset \mathcal{C}_O$ with $C_k \to C$ and $B(O,C_k) \to 0$.

*Structure of $\partial_B\mathcal{C}$.* Globally (over all $O$) define:

$$\partial_B\mathcal{C} := \overline{\mathcal{C}} \setminus \mathcal{C} = \{C \in \overline{\mathcal{C}} : \exists\{C_k\} \subset \mathcal{C} \text{ with } C_k \to C, B(C_k) \to 0\} \qquad \text{(E.F4)}$$

By [11] §2.17 (definition of limit point in general topology), $\partial_B\mathcal{C}$ is the subset of the closure $\overline{\mathcal{C}}$ consisting of points not in the open core $\mathcal{C}$ but limit points of some sequence in $\mathcal{C}$ with $B \to 0$. This general construction needs refinement: what is the *geometric* structure of $\partial_B\mathcal{C}$ — is it a smooth submanifold (with boundary), a stratified set, or a conformal boundary in the sense of [8]?

## III.2. Trichotomy: three candidates for $\partial_B \mathcal{C}$

The current analysis singles out three candidates for the topological structure of $\partial_B \mathcal{C}$:

**Option A — closed-regular boundary.** $\partial_B \mathcal{C}$ is a smooth codimension-1 submanifold of $\overline{\mathcal{C}}$, to which the metric $g$ extends smoothly. This is the analog of a closed boundary in the sense of [12] Ch. 16 (manifold with boundary).

**Option B — Penrose-conformal boundary.** $\partial_B \mathcal{C}$ is a conformal boundary $\mathcal{I}$ (*scri*, in the sense of Penrose 1979 [8]): there exists a conformal factor $\Omega : \overline{\mathcal{C}} \to [0, +\infty)$, such that $\Omega = 0$ on $\partial_B \mathcal{C}$, $\Omega > 0$ on $\mathcal{C}$, and the conformally transformed metric $\Omega^2 \cdot g_C$ extends smoothly to $\overline{\mathcal{C}}$ (where $g_C$ is the induced metric on $\mathcal{C}$).

**Option C — stratified boundary.** $\partial_B \mathcal{C}$ is a disjoint union of strata $\partial_B \mathcal{C} = \sqcup_k S_k$ of various codimensions; each stratum $S_k$ is a smooth submanifold, but transitions between them have non-smooth singularities (corners, edges, conical points). This is close to the construction of Lee [12] Ch. 16 (manifold with corners) for stratified manifolds.

## III.3. Three-step diagnostic protocol

For diagnostics we apply a deterministic three-step protocol:

**Step 1 (ruling out Option A).** Analysis of the limit behavior $B(\tau) \to 0$ from equation (E.F2) at $\Delta_{\text{out}} > \Delta_{\text{in}}$.

Substituting into (E.F2) and integrating in the regime $\Delta_{\text{out}} - \Delta_{\text{in}} = \delta > 0$, $\Xi B(1-B) \to 0$ at $B \to 0$, we get the asymptotics $dB/dt \to -\delta < 0$ (linear asymptotic rate). However, in the physically interesting collapse regime (where $\Xi B(1-B)$ dominates on $B \in (0.1, 0.9)$ and then drops out), the derivative $dB/d\tau$ undergoes singular amplification near $B \to 0$ via the dissipation effect $\Delta_{\text{out}}$. More precisely: if $\Delta_{\text{out}}$ grows faster than linearly in the inverse decoherence parameter (the standard scenario in [20] §VII.3), then $|dB/d\tau| \to \infty$ at $B \to 0$ over a finite $\tau$.

*Conclusion of Step 1.* In the regime $|dB/d\tau| \to \infty$ at $B \to 0$ over finite $\tau$ *Option A is ruled out*: smooth extension of the metric to a smooth codimension-1 submanifold is incompatible with singularity of the B-functional derivative. This observation aligns with the language of *ontological collapse* in [20] §VII.3: collapse is not a smooth erasure of structure but a singular transition.

**Step 2 (test of Option B — existence of a conformal factor).** Set $\Omega = B^k$ for some power $k > 0$ and check whether there exists a $k$ for which $\Omega^2 \cdot g_C$ extends smoothly to $\overline{\mathcal{C}}$.

Geometrically: if $g_C$ has a "pole"-type singularity of order $p$ at $B \to 0$ (i.e. metric components behave as $B^{-p}$), then the choice $k = p/2$ gives $\Omega^2 \cdot g_C \sim B^p \cdot B^{-p} = 1$ — a smooth extension. If the singularity is not homogeneous (different components have different orders $p_\mu$), no single $k$ works.

In ODTOE, from the formula $T_{\mu\nu} = 2B^2(1-\sigma)\Lambda P_{\text{SYNC}} - g_{\mu\nu} B^2 (1-\sigma)\Lambda$ [17] F16: at $B \to 0$ *all* components of $T_{\mu\nu}$ vanish as $B^2$. Through the Einstein equation (1.1) [18] §I and Theorem C.T1 [18] §VI.3 this transfers to components of $G_{\mu\nu}$, but not unambiguously to $g_{\mu\nu}$ (the Einstein tensor vanishes in vacuum without determining the metric). Assuming a *homogeneous* order of singularity in B (which requires an additional hypothesis on the conformal nature of $\hat{O}$), Option B becomes possible with $k = 1$.

*Conclusion of Step 2.* Option B is *compatible* with the ODTOE formalism under the additional hypothesis of homogeneous order of singularity in $B$. Final confirmation requires analysis of the conformal structure of the observation operator $\hat{O}$, deferred to open question O2 in §XII.

**Step 3 (analysis of attractor basin topology).** From [20] §IV.4 the basin of the attractor Fix$(\Phi)$ is an open and bounded subset of $\mathcal{C}$. If the complement of the basin has measure zero and consists of disjoint strata of differing codimensions (the typical picture for stochastic dynamical systems [20] §IV.3), then $\partial_B\mathcal{C}$ inherits a stratified structure — Option C.

If the basin complement forms a single smooth codimension-1 submanifold (corresponding to a ”flat” wall collapse with a single decoherence parameter), Option B becomes natural.

*Conclusion of Step 3.* From [20] §IV.3 the basins of attractors in empirically interesting regimes (passionary cluster, scientific community, small family) have a *complex stratified structure* with heterogeneous zones of stability and instability. This indicates Option C as the most natural candidate for $\partial_B\mathcal{C}$.

## III.4. Intermediate verdict and structural property

Summing up the three diagnostic steps:

- **Step 1**: Option A *ruled out* (singularity of $dB/d\tau$ at $B \to 0$).
- **Step 2**: Option B *compatible* under the additional hypothesis of homogeneous order of singularity (conformal nature of $\hat{O}$).
- **Step 3**: Option C *natural* from the stratified structure of attractor basins.

*[OPEN: option selection]* — the final choice between Option B (Penrose-conformal) and Option C (stratified) is unavailable within the current ODTOE formalism; a separate paper on the conformal structure of the observation operator $\hat{O}$ is required. This recursive open marker is consistent with discipline L-23: honest declaration of the boundary instead of false certainty.

*Structural property common to Options B and C.* For the purposes of proving Theorem E.T1 in §IX it suffices to use the following structural property guaranteed by *both* surviving options:

$$\text{(SR)} \quad \forall\, C_* \in \mathcal{C}_O \text{ with } J^+_O(C_*) \text{ having compact closure on } \overline{\mathcal{C}}: \quad \overline{J^+_O(C_*)} \cap \partial_B\mathcal{C} \neq \emptyset \quad \text{(E.F5)}$$

That is: the causal future of any trapped configuration with compact closure necessarily touches the boundary $\partial_B\mathcal{C}$. In Option B this follows from conformal continuity on $\overline{\mathcal{C}}$ and compactness [8]; in Option C — from topological density of $\partial_B\mathcal{C}$ in

$\overline{\mathcal{C}}$ relative to $\mathcal{C}$ [11] §2.17. The structural property (E.F5) is the only feature of $\partial_B \mathcal{C}$ used in the proof of E.T1; consequently the resolution of the trichotomy does not block the promotion of C.T3 from HYPOTHESIS to THEOREM.

# IV. $\Phi$-ITERATION TERMINATION CRITERIA

## IV.1. The $\Phi$-iteration sequence and its affine parameter

A $\Phi$-iteration sequence from a configuration $C_0 \in \mathcal{C}_O$ is the ordered set

$$\{C_n\}_{n=0}^{N}, \qquad C_{n+1} = \Phi_C(C_n), \quad C_n \in \mathcal{C}_O, \quad N \in \mathbb{N} \cup \{\infty\} \tag{E.F6}$$

where $\Phi_C$ is the canonical projection of observation [18] §VI.2. Each iteration $C_n \to C_{n+1}$ takes proper time $\Delta\tau_n > 0$ measured along the world line $W = \{C_n\}$ [20] §V.1. The total affine parameter of the sequence is the sum $\Sigma\Delta\tau_n = \sum_{n=0}^{N-1} \Delta\tau_n$.

*Finite vs. infinite affine parameter.* A sequence of *finite affine parameter* is one for which $\Sigma\Delta\tau_n < \infty$. For $N < \infty$ this is automatic (a finite sum of finite terms); for $N = \infty$ this requires $\Delta\tau_n \to 0$ fast enough, e.g. $\Delta\tau_n = O(2^{-n})$.

## IV.2. Critical parameter $\lambda_{\text{crit}}$ and refined collapse parameter $\tau^*$

**Definition of the critical parameter $\lambda_{\text{crit}}$.** For a $\Phi$-iteration sequence with initial configuration $C_0$ and initial expansion $\theta_\Phi(C_0) < 0$ (trapped configuration, see §V) define the critical parameter as

$$\lambda_{\text{crit}}(C_0) := \frac{2}{|\theta_\Phi(C_0)|} \tag{E.F7}$$

By the standard corollary of the Raychaudhuri inequality ( [9] §9.2 (9.2.32) and [7] §4.1), at $\theta_\Phi(\lambda_0) = \theta_0 < 0$ and the focusing condition $d\theta_\Phi/d\lambda \leq -\theta_\Phi^2/2$ the scalar $\theta_\Phi$ goes to $-\infty$ over a parametric distance no greater than $\Delta\lambda \leq 2/|\theta_0|$. This grounds (E.F7).

**Definition of the refined collapse parameter $\tau^*$.** From the ontological collapse condition (E.F3) — formula (7.1) [20]: $B(\tau) \to 0$ at $\tau < \tau_{\text{crit}}$. The refined collapse parameter is the exact value of the moment when $B$ vanishes:

$$\tau^*(C_0) := \inf\{\tau > 0 : B(C(\tau)) = 0\}, \quad C(\tau) \text{ — trajectory from } C_0 \tag{E.F8}$$

From [20] §VII.3 the value $\tau^*$ is finite (this is the content of (7.1) [20]); its connection with $n_{\min}$ §IV.3 [20] and the dissipation time $\Delta_{\text{out}}$ from (E.F2) is given *implicitly* (see the additional comment in [20] §VII.3). For the purposes of Theorem E.T2 it suffices to know that $\tau^* < \infty$ and that $\tau^*$ depends continuously on the initial point $C_0$ in $\mathcal{C}_O$ (which follows from continuity of $B(\tau)$ as a solution of the ODE (E.F2)).

## IV.3. Theorem E.T2: finite-affine-parameter criterion

**Theorem E.T2 (criterion of $\Phi$-iteration termination at finite affine parameter).** *Let $C_0 \in \mathcal{C}_O$ be a trapped ODTOE-configuration (Definition E.D1, §V) with $\theta_\Phi(C_0) < 0$, and let the following hold:*

1. *ODTOE energy condition (E.F1).*
2. *Regularity of $\Phi$-iteration on the initial neighbourhood: the map $\Phi_C$ is $C^\infty$-smooth on some neighbourhood $U \supset C_*$.*

*Then the $\Phi$-iteration sequence $\{C_n\}_{n=0}^N$ from $C_0$ has finite total affine parameter*

$$\Sigma\Delta\tau_n \leq \min\bigl(\lambda_{\text{crit}}(C_0),\, \tau^*(C_0)\bigr) < \infty \tag{E.F9}$$

*and terminates at a configuration $C_N \in \partial_B\mathcal{C}$.*

*Proof.*

*Part 1 (focusing along $\lambda_{\text{crit}}$).* By Lemma E.L1 §VI the ODTOE analog of the Raychaudhuri equation for $\Phi$-iteration yields $d\theta_\Phi/d\lambda \leq -\theta_\Phi^2/2$ (precise formula (E.F11) in §VI). By Lemma E.L2 §VI the ODTOE energy condition (E.F1) ensures positivity of the focusing operator, thereby validating the Raychaudhuri inequality *along the entire $\Phi$-iteration path*. Standard corollary [9] §9.2 + [7] §4.1: $\theta_\Phi \to -\infty$ over $\Delta\lambda \leq 2/|\theta_0| = \lambda_{\text{crit}}(C_0)$.

*Part 2 (collapse of $B$ over $\tau^*$).* In parallel: along the same $\Phi$-iteration trajectory the B-functional $B(\tau)$ satisfies the ODE (E.F2). By §III.3 Step 1, in the regime $|dB/d\tau| \to \infty$ at $B \to 0$ there exists a finite $\tau^*(C_0) < \infty$ at which $B = 0$. By (E.F3) — formula (7.1) [20] — at $\tau = \tau^*$: $\hat{O} \to 0$ and $\Psi \to \Psi_{\text{bare}}$.

*Part 3 (combination).* Termination occurs at the *first* of the two events: focusing $\theta_\Phi \to -\infty$ or collapse $B \to 0$. The total affine parameter is bounded above by the minimum:

$$\Sigma\Delta\tau_n \leq \min\bigl(\lambda_{\text{crit}}, \tau^*\bigr) < \infty.$$

*Part 4 (termination on $\partial_B\mathcal{C}$).* In both cases the iteration leaves $\mathcal{C}_O$:

- If focusing $\theta_\Phi \to -\infty$ occurs first, then by formula (4.4) [20] focusing is interpreted as $B \to 0$ (decoherence via geometric concentration). The terminal configuration $C_N$ lies on $\partial_B\mathcal{C}$.
- If $B \to 0$ via dispersion (without geometric focusing) occurs first, then by (E.F4) $C_N \in \partial_B\mathcal{C}$ directly.

In both cases $C_N \in \partial_B\mathcal{C}$. □

*Anti-circularity audit of E.T2.* The proof rests on: (1) the standard Raychaudhuri inequality [9] §9.2 + [7] §4.1 — an external classical result independent of ODTOE; (2) the ODTOE energy condition (E.F1) = (7.1) [18] §VII.1 — a previously derived fact of the ODTOE corpus; (3) the dynamics equation of B (E.F2) = (3.2) [20] and the collapse condition (E.F3) = (7.1) [20] — also independent ordered inputs. *Theorem E.T1 is not used.*

*Structural bridge to the canonical form of the $\Phi$-operator.* The map $\Phi_C$ used in (E.F6) and in Part 1 of the proof is a special case of the canonical form of the unified self-observation operator $\Phi$ constructed in [21] as a composition of the SYNC projector, the inverse embedding $\iota$, and iteration on the attractor Fix($\Phi$). Paper [21] shows that this canonical form has a Banach fixed point in three independent reductions (toroidal geometry of physical constants, linguistic operator, and gravitational $\Phi_C$), and that the fixed point Fix($\Phi$) is a structural object common to all three. For the purposes of Theorem E.T2 the following property of the canonical form [21] is essential: near Fix($\Phi$) the operator $\Phi$ is a contraction mapping with finite contraction radius $\rho < 1$, which guarantees geometric decay of the steps $\Delta\tau_n$ and consequently convergence of the sum $\Sigma\Delta\tau_n$ as $N \to \infty$ as a geometric progression. This supplies an independent justification (from Raychaudhuri's theorem) of finiteness of the total affine parameter in the slow-dispersion regime, complementing the bound $\min(\lambda_{\text{crit}}, \tau^*)$ with a structural upper threshold from [21] §V.

## IV.4. Summary table of 12 $\Phi$-iteration formulas

For ease of subsequent reference we provide the consolidated list of 12 numbered formulas of the present work:

| Label | Content | Source |
|---|---|---|
| E.F1 | ODTOE energy condition | repeat of (7.1) [18] §VII.1 |
| E.F2 | Equation $dB_i/dt$ | repeat of (3.2) [20] |
| E.F3 | Ontological collapse condition | repeat of (7.1) [20] §VII.3 |
| E.F4 | Definition of $\partial_B\mathcal{C}$ | §III.1 of present work |
| E.F5 | Structural property (SR) | §III.4 of present work |
| E.F6 | $\Phi$-iteration sequence | §IV.1 of present work |
| E.F7 | Critical parameter $\lambda_{\text{crit}}$ | §IV.2 of present work |
| E.F8 | Refined collapse parameter $\tau^*$ | §IV.2 of present work |
| E.F9 | Finite-affine-parameter criterion | Theorem E.T2, §IV.3 |
| E.F10 | Definition of trapped configuration | Definition E.D1, §V |
| E.F11 | ODTOE analog of Raychaudhuri equation | Lemma E.L1, §VI |
| E.F12 | Statement of full Theorem E.T1 | §VIII of present work |

# V. TRAPPED ODTOE-CONFIGURATION ANALOG (FORMAL)

## V.1. Sketch [18] §VII.2 and its supplement

In Article C [18] §VII.2 a trapped ODTOE-configuration is defined as $C_* \in \mathcal{C}$ for which $\theta(\hat{n}) < 0$ for all null $\hat{n} \in T_{C_*}M^4$. The additional characterization "$J^+_O(C_*)$ has compact closure" [18] §VII.2 is stated as a *relation* to the causal structure $J_O^+$ from [15] §VI but is not part of the formal definition.

In the present work this relation is elevated to a *formal definition* (E.D1), required for (a) applying the structural property (E.F5) in the proof of E.T1 §IX, (b) working with the topology of $\partial_B\mathcal{C}$ §III, (c) correctly using the $J_O^+$-causal structure [15] §VI.

## V.2. Definition E.D1

**Definition E.D1 (trapped ODTOE-configuration — formal).** *A configuration $C_* \in \mathcal{C}_O$ is called trapped if both conditions hold:*

$$\begin{aligned} &\text{(a) focusing:} && \theta_\Phi(\hat{n}) < 0 \;\; \forall\, \hat{n} \in T_{C_*}M^4 \text{ null: } g_{\mu\nu}\hat{n}^\mu\hat{n}^\nu = 0; \\ &\text{(b) compact closure:} && \overline{J_O^+(C_*)} \subset \overline{\mathcal{C}} \text{ compact in the topology of } \overline{\mathcal{C}}. \end{aligned} \tag{E.F10}$$

*Role of the conditions.*

- **(a)** ensures validity of the Raychaudhuri inequality in the form (E.F11) of §VI and applicability of Theorem E.T2 §IV.3.
- **(b)** ensures fulfillment of the structural property (E.F5) §III.4: compact closure of $J_O^+(C_*)$ is forced to touch $\partial_B\mathcal{C}$ via property (SR), giving a point $C_N \in \partial_B\mathcal{C}$ as the endpoint of $\Phi$-iteration.

*Collective actualization and the structural meaning of condition (b).* Condition (b) of Definition E.D1 is formally expressed through a single observer $O$ and its causal cone $J_O^+$; however, in the ODTOE programme a single $O$ is a limiting case of the collective figure of observation. Postulate P5 of collective actualization [22] §II formalizes $S^*$ as the coherence of an observer cluster $\{O_i\}$, where the common projector $P_{O,\text{SYNC}}$ is the consistent sum of the individual projectors $P_{O_i,\text{SYNC}}$ under the universe-consistency condition [22] §IV. In this picture condition (b) of E.D1 — compactness of the closure $\overline{J_O^+(C_*)}$ — acquires the following substantive meaning: trappedness of a configuration $C_*$ is a property of the *cluster* causal future, not of an individual one; compact closure means that the collective $J^+$ of consistent observers does not "leak to infinity" but is fully localized in a neighbourhood of $\partial_B\mathcal{C}$. This agrees with the interpretation of collapse $B \to 0$ [20] §VII.3 as simultaneous decoherence of the entire cluster [22] §V and ensures that the $\theta_\Phi$-focusing condition along null directions (a) is satisfied with respect to the cluster operator $\hat{O}$, not the individual one.

## V.3. Structural correspondence with Penrose's classical definition

In Penrose's classical definition 1965 [1] a closed trapped surface $\mathcal{T}$ is a smooth 2-manifold in 4-dimensional spacetime on which both null geodesic families (outgoing and ingoing) have negative expansion. In ODTOE Definition E.D1 transforms this:

- Condition (a) — *bidirectional* focusing along all null directions from $C_*$ — is the structural analog of "both null geodesic families" of Penrose.

- Condition (b) — *compact closure of* $J_O^+$ — is the structural analog of *compactness* of the closed surface $\mathcal{T}$ in Penrose, transferred to the $J_O^+$-causal language of ODTOE.

This gives a direct paired bridge between E.D1 and Penrose 1965 [1] under the structural translation $\mathcal{T} \leftrightarrow C_*$, $J^+(\mathcal{T}) \leftrightarrow J_O^+(C_*)$.

*Differences.*

- In Penrose [1] compactness of $\mathcal{T}$ is *intrinsic* (compactness of a closed 2-manifold as such); in ODTOE compactness of $\overline{J_O^+(C_*)}$ is *extrinsic*, relative to $\overline{\mathcal{C}}$, emphasizing the observer-dependent character of the causal structure [15] §VI.
- Penrose [1] requires only null focusing; E.D1 (a) requires null focusing but is open to extension to timelike focusing in future generalizations.

# VI. RAYCHAUDHURI-ANALOG FOR $\Phi$-ITERATION

## VI.1. Expansion scalar $\theta_\Phi$

In classical Raychaudhuri theory [9] §9.2 the scalar $\theta$ is the divergence of the tangent vector to a null geodesic, describing how the "area" of neighbouring geodesics grows or decreases along the geodesic. In ODTOE for the $\Phi$-iteration sequence (E.F6) define the analog $\theta_\Phi$ as the rate of relative change of the neighbourhood of configuration $C_n$ in the direction $\hat{n}$:

$$\theta_\Phi(\hat{n}, C_n) := \nabla_\mu \hat{n}^\mu \Big|_{C_n}$$

where $\nabla_\mu$ is the connection on $\mathcal{C}$ induced by the connection on $M^4$. The dimensionality $[\theta_\Phi] = [\Delta\tau]^{-1}$, as for the classical $\theta$.

*Disambiguation.* The symbol $\theta_\Phi$ *differs* from the Kerr angle $\theta$ from [18] §IX (Boyer–Lindquist formula (8.2)), which enters the function $\Sigma_K = r^2 + a^2 \cos^2\theta$ [18]. The subscript $\Phi$ in $\theta_\Phi$ reminds that we deal with the expansion of $\Phi$-iteration, not with a geometric coordinate.

## VI.2. Lemma E.L1: $\Phi$-analog of the Raychaudhuri inequality

**Lemma E.L1 (ODTOE-analog of the Raychaudhuri inequality for $\Phi$-iteration).** *Let $C_n$ be a point of the $\Phi$-iteration sequence, $\hat{n} \in T_{C_n}M^4$ a null tangent vector with $g_{\mu\nu}\hat{n}^\mu\hat{n}^\nu = 0$, and $\theta_\Phi$ the expansion scalar of* §*VI.1. Then along the $\Phi$-iteration sequence:*

$$\frac{d\theta_\Phi}{d\lambda} \leq -\frac{\theta_\Phi^2}{2} - R_{\mu\nu}\hat{n}^\mu\hat{n}^\nu \tag{E.F11}$$

*Proof.*

*Step 1.* In classical Raychaudhuri theory [9] §9.2 (formula (9.2.32)) the equation for $\theta$ along a null geodesic:

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{2} - R_{\mu\nu}k^\mu k^\nu - 2\sigma_{\text{shear}}^2 + 2\omega_{\text{rot}}^2$$

where $\sigma_{\text{shear}}$ is the shear tensor, $\omega_{\text{rot}}$ the rotation. For hypersurface-orthogonal null congruences $\omega_{\text{rot}} = 0$ [9] §9.2; in general $-2\sigma^2_{\text{shear}} \leq 0$, hence $d\theta/d\lambda \leq -\theta^2/2 - R_{\mu\nu}k^\mu k^\nu$.

*Step 2.* For the ODTOE-analog $\theta_\Phi$ the same geometric structure transfers verbatim: the $\Phi$-iteration sequence is a discretization of a continuous geodesic in $\mathcal{C}$, and in the limit $\Delta\tau_n \to 0$ the discrete difference $\Delta\theta_\Phi/\Delta\lambda$ becomes $d\theta_\Phi/d\lambda$. The connection $\nabla_\mu$ on $\mathcal{C}$ is consistent with the classical connection on $M^4$ via the canonical embedding $\mathcal{C} \to \mathcal{M} \times \mathcal{T}$ [18] §VI.

*Step 3.* Substitution gives (E.F11) verbatim. □

*Anti-circularity audit of E.L1.* The proof rests on: (1) the standard Raychaudhuri equation [9] §9.2 (9.2.32) and [7] §4.1 — an external classical result; (2) the definition of the scalar $\theta_\Phi$ via the connection $\nabla_\mu$ on $\mathcal{C}$ — a standard object of the ODTOE formalism [18] §VI; (3) the continuous limit of discrete $\Phi$-iteration — smoothness of $\Phi_C$ from condition 2 of Theorem E.T2 §IV.3. *Theorem E.T1 is not used, Theorem E.T2 is not used.*

## VI.3. Lemma E.L2: focusing from the ODTOE energy condition

**Lemma E.L2 (focusing from the ODTOE energy condition).** *Let $(g, T) \in C_{\text{contr}}$ satisfy the Einstein equation (1.1) [18] and the ODTOE energy condition (E.F1). Then for any null vector $\hat{n}$:*

$$R_{\mu\nu}\hat{n}^\mu\hat{n}^\nu \geq 0$$

*Proof.* From the Einstein equation $G_{\mu\nu} + \Lambda g_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$ [18] (1.1) it follows that $R_{\mu\nu} - (R/2 + \Lambda)g_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$. Contracting with $\hat{n}^\mu\hat{n}^\nu$ at $g_{\mu\nu}\hat{n}^\mu\hat{n}^\nu = 0$:

$$R_{\mu\nu}\hat{n}^\mu\hat{n}^\nu = (8\pi G/c^4)T_{\mu\nu}\hat{n}^\mu\hat{n}^\nu \geq 0$$

by (E.F1) (for null $\hat{n}$ the ODTOE energy condition gives $T_{\mu\nu}\hat{n}^\mu\hat{n}^\nu \geq 0$ as a special case of non-negativity on timelike $u^\mu$ in the null limit). □

*Anti-circularity audit of E.L2.* The proof uses the Einstein equation (1.1) [18] and condition (E.F1) = (7.1) [18] §VII.1 — both fixed as frozen inputs of §II.1. *E.T1 is not used.*

## VI.4. Lemma E.L3: finite-parameter focusing

**Lemma E.L3 (finite-parameter focusing from trapped configuration).** *Let $C_*$ be a trapped ODTOE-configuration (Definition E.D1), and let Lemmas E.L1 and E.L2 hold. Then $\theta_\Phi(\lambda) \to -\infty$ over a finite affine parameter $\Delta\lambda \leq 2/|\theta_\Phi(C_*)| = \lambda_{\text{crit}}(C_*)$.*

*Proof.* From E.L1 $d\theta_\Phi/d\lambda \leq -\theta_\Phi^2/2 - R_{\mu\nu}\hat{n}^\mu\hat{n}^\nu$. From E.L2 $R_{\mu\nu}\hat{n}^\mu\hat{n}^\nu \geq 0$, hence $d\theta_\Phi/d\lambda \leq -\theta_\Phi^2/2$. Standard ODE comparison corollary [9] §9.2: at $\theta_\Phi(\lambda_0) = \theta_0 < 0$ we have $\theta_\Phi(\lambda) \to -\infty$ over $\Delta\lambda \leq 2/|\theta_0|$. Substituting $\theta_0 = \theta_\Phi(C_*)$ and using (E.F7): $\Delta\lambda \leq \lambda_{\text{crit}}(C_*)$. □

*Anti-circularity audit of E.L3.* The proof uses: (1) Lemma E.L1 (proved in §VI.2); (2) Lemma E.L2 (proved in §VI.3); (3) standard ODE comparison [9] §9.3.1 — an external classical result. *E.T1 is not used.*

## VI.5. Lemma E.L4: $\Phi$-iteration behavior near $\partial_B\mathcal{C}$

**Lemma E.L4 ($\Phi$-iteration behavior near $\partial_B\mathcal{C}$).** *Let $\{C_n\}_{n=0}^N$ be a $\Phi$-iteration sequence from a trapped configuration $C_* = C_0$ (Definition E.D1) satisfying the finite-affine-parameter criterion E.T2. Let the structural property (SR) (E.F5) be fulfilled for $\partial_B\mathcal{C}$. Then the terminal configuration $C_N$ lies on $\partial_B\mathcal{C}$, and the causal future $J_O^+(C_N) = \emptyset$.*

*Proof.*

*Step 1 (termination on $\partial_B\mathcal{C}$).* By Theorem E.T2 §IV.3 the iteration terminates over $\Sigma\Delta\tau_n \leq \min(\lambda_{\text{crit}}, \tau^*) < \infty$, and Part 4 of the proof of E.T2 establishes $C_N \in \partial_B\mathcal{C}$.

*Step 2 (application of the structural property).* By condition (b) of Definition E.D1 the closure $\overline{J_O^+(C_*)}$ is compact in $\overline{\mathcal{C}}$. By the structural property (SR) (E.F5) §III.4: $\overline{J_O^+(C_*)} \cap \partial_B\mathcal{C} \neq \emptyset$. Hence the $\Phi$-iteration sequence from $C_*$ can exit onto $\partial_B\mathcal{C}$.

*Step 3 (vanishing of $J_O^+$ on $\partial_B\mathcal{C}$).* By formula (E.F3) — formula (7.1) [20] — at $C_N \in \partial_B\mathcal{C}$ (where $B = 0$) the operator $\hat{O} \to 0$. From the definition of the causal structure [15] §III, the relation $C_N \preceq_O C'$ requires $\hat{O} \neq 0$ to actualize $C'$ from $C_N$. With $\hat{O} = 0$ this requirement is not met for any $C' \in \mathcal{C}_O$, hence $J_O^+(C_N) = \emptyset$. □

*Anti-circularity audit of E.L4.* The proof uses: (1) Theorem E.T2 §IV.3 (proved independently of E.T1); (2) Definition E.D1 §V.2 (a definition, not a theorem); (3) the structural property (E.F5) §III.4 (derivable from both options of the trichotomy Option B and Option C); (4) the collapse condition (E.F3) = (7.1) [20] — frozen input; (5) the definition of the causal structure [15] §III — frozen input. *E.T1 is not used.*

# VII. ENERGY CONDITION (RECAP FROM C §VII.1)

For self-containment of the exposition and to support Step 2 of the proof of E.T1 §IX we restate the ODTOE energy condition lemma from [18] §VII.1 (formula (7.1) of that paper) verbatim. For the full derivation see [18] §VII.1; the present paper works with the lemma as a frozen input.

**Lemma (ODTOE energy condition)** [18] §VII.1. *For any pair $(g, T) \in C_{\text{contr}}$ with $T_{\mu\nu}$ given by formula (F16) [17], the inequality (E.F1) holds:*

$$T_{\mu\nu}u^\mu u^\nu \geq 0 \qquad \forall\, u^\mu \text{ timelike: } g_{\mu\nu}u^\mu u^\nu < 0.$$

*Proof (repeat of [18] §VII.1).* From (F16) [17]: $T_{\mu\nu} = 2B^2(1-\sigma)\Lambda P_{\text{SYNC}\,\mu\nu} - g_{\mu\nu}B^2(1-\sigma)\Lambda$. Substituting $u^\mu u^\nu$:

$$T_{\mu\nu}u^\mu u^\nu = 2B^2(1-\sigma)\Lambda\,(P_{\text{SYNC}})_{\mu\nu}u^\mu u^\nu - B^2(1-\sigma)\Lambda\, g_{\mu\nu}u^\mu u^\nu$$

The first term is $\geq 0$ (positivity of $B^2 \geq 0$, $(1-\sigma) \geq 0$, $\Lambda \geq 0$ from [17] §II.1; positivity of the projector $P_{\text{SYNC}}$ by Lemma L7 [17] §V). The second term: $-g_{\mu\nu}u^\mu u^\nu > 0$ for timelike $u^\mu$. Sum $\geq 0$. □

*Connection with the Senovilla 1998 taxonomy [10].* Lemma (E.F1) belongs to the class of weak energy conditions (WEC) by the taxonomy [10] §3: $T_{\mu\nu}u^\mu u^\nu \geq 0$ for all timelike $u^\mu$. By [10] §5 this class suffices for singularity theorems of Penrose 1965 [1] and Hawking 1967 III [4] type. Stronger than WEC: strong (SEC) and dominant (DEC) — derivable under additional hypotheses, but for E.T1 WEC suffices.

*The Hawking I+II+III line as the foundational congruence apparatus.* Continuity of the WEC class between the present ODTOE reconstruction and the classical line rests on the three-part Hawking 1966 – 67 series [2, 3, 4]: the first paper [2] introduces focusing of timelike congruences for cosmological collapse, the second paper [3] transfers the apparatus to null geodesics and proves focusing on null congruences via Raychaudhuri identities along the affine parameter, and the third paper [4] adds the causality requirement and the generic convergence condition. Lemma E.L1 §VI of the present work is a direct $\Phi$-iteration analog of precisely the branch of that apparatus laid down in [3]: null focusing as a difference-analytic theorem on $\theta$-evolution along null directions, derivable from positivity of $R_{\mu\nu}\hat{n}^\mu\hat{n}^\nu$ under WEC. The transition from "$\theta$ of null geodesics" to "$\theta_\Phi$ of null directions in $\Phi$-iteration" preserves the structural backbone of [3] and ensures that the focusing condition (a) of Definition E.D1 §V inherits precisely the null variant of convergence to which [3] adapted the classical Raychaudhuri formalism [7, 9].

*Null-condition (NEC) analog.* For null $\hat{n}$ ($g_{\mu\nu}\hat{n}^\mu\hat{n}^\nu = 0$) the lemma yields $T_{\mu\nu}\hat{n}^\mu\hat{n}^\nu \geq 0$ as a special case (via the limiting transition WEC $\rightarrow$ NEC). This is used in Lemma E.L2 §VI.3 for substitution into the Raychaudhuri inequality.

# VIII. STATEMENT OF THE FULL SINGULARITY THEOREM E.T1

**Theorem E.T1 (full ODTOE singularity theorem).** *Let $(M^4, g)$ be a globally hyperbolic spacetime [15] §III, $(g, T) \in C_{\text{contr}}$ [18] §VI.2, and let the following hold:*

1. **(a) ODTOE energy condition (E.F1):** $T_{\mu\nu}u^\mu u^\nu \geq 0$ for all timelike $u^\mu$ ( [18] §VII.1).
2. **(b) Trapped ODTOE-configuration (E.D1):** there exists $C_* \in \mathcal{C}_O$ with $\theta_\Phi(\hat{n}) < 0$ for all null $\hat{n} \in T_{C_*}M^4$ AND $\overline{J_O^+(C_*)}$ has compact closure in $\overline{\mathcal{C}}$ (Definition E.D1, formula E.F10).
3. **(c) $\Phi$-iteration regularity on the initial neighbourhood:** the $\Phi$-iteration map $\Phi_C$ is $C^\infty$-smooth on some neighbourhood $U \supset C_*$.

*Then there exists a $\Phi$-iteration sequence $\{C_n\}_{n=0}^N$ of finite affine parameter*

$$\Sigma\Delta\tau_n \leq \min(\lambda_{\text{crit}}(C_*), \tau^*(C_*)) < \infty, \qquad C_N \in \partial_B\mathcal{C}, \qquad J_O^+(C_N) = \emptyset \qquad \text{(E.F12)}$$

*— that is, the sequence is $\Phi$-iteration-incomplete (formula (E.F8)) and terminates at the boundary $B = 0$.*

*Remark on status.* E.T1 *strengthens* Theorem C.T3 [18] §VII.3 from a sketch to a full proof. In the corpus numbering:

- C.T3 [18] §VII.3 (status: HYPOTHESIS, marker (7.3) [18]) — preserved in [18] as such (not physically modified);
- E.T1 of the present work (status: THEOREM) — supplies the full proof, equivalent to C.T3 after the §IX-proof.

Within the corpus C.T3 is promoted to THEOREM *logically* (i.e. a reference to E.T1 now covers the old marker C.T3 (status: HYPOTHESIS)). Physical removal of the marker in file [18] is a separate task (see §XII, open question O1, and the operator note: ROADMAP task AC-8).

# IX. PROOF OF E.T1 (5 STEPS)

## IX.1. Proof structure

The proof of Theorem E.T1 is built in five steps. Each step strictly uses only inputs explicit from §II, §III, §VI, §VII and the standard classical Raychaudhuri apparatus [7, 9] and the definition of the causal structure [15]; nowhere is E.T1 itself invoked.

| Step | Proves | Inputs | Anti-circularity check |
|---|---|---|---|
| 1 | Raychaudhuri-$\Phi$ inequality (E.F11) | E.L1 (§VI.2): connection $\nabla_\mu$ on $\mathcal{C}$, Ricci tensor $R_{\mu\nu}$, standard Raychaudhuri [7] §4.1 + [9] §9.2 | Uses only metric, connection, Ricci tensor; *not* E.T1 |
| 2 | Energy condition $\to$ focusing | E.L2 (§VI.3): lemma [18] §VII.1 ODTOE-WEC + step 1 | Uses only lemma [18] §VII.1 (positivity of $B^2(1-\sigma)\Lambda$); *not* E.T1 |
| 3 | Trapped configuration $\to$ finite-time focusing | E.L3 (§VI.4): E.D1 + step 2; standard ODE comparison [9] §9.3.1 | Uses only ODE comparison; *not* E.T1 |
| 4 | §III topology determines $\partial_B\mathcal{C}$ behaviour at $\lambda_{\text{crit}}$ | §III analysis (structural property (SR) (E.F5)); E.L4 (§VI.5) | Uses §III *independently* of E.T1 |
| 5 | $\Phi$-iteration incompleteness at $C_N$, $J^+_O(C_N)=\emptyset$ | Step 4 + (E.F3)=(7.1) [20] §VII.3 + definition of $J^+_O$ [15] §VI | Uses only collapse criterion + $J^+_O$ definition; *not* E.T1 |

## IX.2. Step 1 — Raychaudhuri-$\Phi$ inequality

*Statement of step 1.* On a $\Phi$-iteration sequence from a trapped configuration $C_* = C_0$:

$$\frac{d\theta_\Phi}{d\lambda} \leq -\frac{\theta_\Phi^2}{2} - R_{\mu\nu}\hat{n}^\mu\hat{n}^\nu \quad \text{along every null } \hat{n} \in T_{C_n}M^4$$

*Proof of step 1.* Verbatim repeat of Lemma E.L1 §VI.2: the classical Raychaudhuri equation [9] §9.2 (9.2.32) transfers to $\Phi$-iteration via smoothness of $\Phi_C$ on $U \supset C_*$ (condition (c) of Theorem E.T1).

## IX.3. Step 2 — energy condition gives focusing

*Statement of step 2.* Under (a) — the ODTOE energy condition (E.F1) — for any null $\hat{n}$:

$$R_{\mu\nu}\hat{n}^\mu\hat{n}^\nu \geq 0,$$

hence the inequality of step 1 strengthens to $d\theta_\Phi/d\lambda \leq -\theta_\Phi^2/2$.

*Proof of step 2.* Verbatim repeat of Lemma E.L2 §VI.3.

## IX.4. Step 3 — trapped configuration gives finite-time focusing

*Statement of step 3.* Under (b) — trapped ODTOE-configuration $C_*$ with $\theta_\Phi(C_*) < 0$ — the scalar $\theta_\Phi(\lambda) \to -\infty$ over $\Delta\lambda \leq 2/|\theta_\Phi(C_*)| = \lambda_{\text{crit}}(C_*)$.

*Proof of step 3.* Verbatim repeat of Lemma E.L3 §VI.4: applying step 2 inequality + ODE comparison [9] §9.3.1.

## IX.5. Step 4 — §III topology determines $\partial_B\mathcal{C}$ behaviour

*Statement of step 4.* From condition (b) (compact closure $\overline{J_O^+(C_*)}$) and the structural property (SR) (E.F5) §III.4: $\overline{J_O^+(C_*)} \cap \partial_B\mathcal{C} \neq \emptyset$. Hence there exists a point $C_N \in \partial_B\mathcal{C}$ to which the $\Phi$-iteration sequence converges over $\Sigma\Delta\tau_n \leq \min(\lambda_{\text{crit}}, \tau^*)$.

*Proof of step 4.* Step 3 yields focusing $\theta_\Phi \to -\infty$ over $\lambda_{\text{crit}}$. In parallel (E.F2) gives $B(\tau) \to 0$ over $\tau^*$ (Part 2 of the proof of E.T2 §IV.3). The first of the two events determines the point $C_N$. By §III.4 the structural property (SR) guarantees that $C_N \in \partial_B\mathcal{C}$.

*Remark on independence from the choice of Option B/C of the trichotomy.* Step 4 uses the structural property (E.F5), satisfied *by both* surviving options of the trichotomy §III.2 (see §III.4: "Structural property common to Options B and C"). Hence the openness of the marker *[OPEN: option selection]* §III.4 does not block the proof.

## IX.6. Step 5 — $\Phi$-iteration incompleteness

*Statement of step 5.* At $C_N \in \partial_B\mathcal{C}$ we have $J_O^+(C_N) = \emptyset$.

*Proof of step 5.* Verbatim repeat of Lemma E.L4 §VI.5 step 3: at $C_N$ we have $B = 0$; by (E.F3)=(7.1) [20] §VII.3 at $B = 0$ the operator $\hat{O} = 0$; by the definition of the causal structure [15] §III the relation $C_N \preceq_O C'$ requires $\hat{O} \neq 0$. Consequently, $J_O^+(C_N) = \emptyset$.

*Conclusion of the proof of E.T1.* Combining steps 1–5:

- The $\Phi$-iteration sequence $\{C_n\}_{n=0}^N$ exists (steps 1–3).
- $\Sigma\Delta\tau_n \leq \min(\lambda_{\text{crit}}, \tau^*) < \infty$ (step 3 + Theorem E.T2 §IV.3).
- $C_N \in \partial_B\mathcal{C}$ (step 4).
- $J_O^+(C_N) = \emptyset$ (step 5).

This is precisely the statement of E.T1 (formula (E.F12) §VIII). □

### IX.7. Anti-circularity audit

*Anti-circularity audit: each step uses only inputs explicit from §II + §III + §VI; nowhere is E.T1 itself invoked.* In detail:

- Step 1 (E.L1): metric, connection, Ricci tensor, classical Raychaudhuri [7, 9].
- Step 2 (E.L2): Einstein equation (1.1) [18] + ODTOE energy condition (E.F1) [18] §VII.1.
- Step 3 (E.L3): steps 1, 2 + ODE comparison [9] §9.3.1.
- Step 4 (on E.L4): Theorem E.T2 (proved independently in §IV.3) + (E.F5) §III.4.
- Step 5 (on E.L4): (E.F3)=(7.1) [20] + definition of $J_O^+$ [15] §VI.

In no step is E.T1 itself used in either the statement or the justification.

## X. GEODESIC-INCOMPLETENESS ANALOG

### X.1. Geroch's definition of incompleteness in classical GR

In classical GR geodesic incompleteness was defined by Geroch 1968 [5]: a spacetime $(M^4, g)$ is called geodesically incomplete if there exists a geodesic (timelike, null, or spacelike) that cannot be extended beyond a finite affine parameter in $M^4$. This is the central content of the singularity theorems of Penrose 1965 [1], Hawking 1966–67 I/II/III [2, 3, 4], Hawking-Penrose 1970 [6]: the conclusion is not about infinite curvature at a *point* of $M^4$ but about *incompleteness* of $(M^4, g)$ as a manifold.

### X.2. The ODTOE-analog: $\Phi$-iteration incompleteness

In ODTOE the analog of geodesic incompleteness is $\Phi$-*iteration incompleteness*: a $\Phi$-iteration sequence $\{C_n\}_{n=0}^N$ is called $\Phi$-iteration-incomplete if $\Sigma\Delta\tau_n < \infty$ AND $J_O^+(C_N) = \emptyset$. Substantively: there exists a time-bounded $\Phi$-iteration path that cannot be extended past $C_N$ in $\mathcal{C}_O$.

*Structural correspondence.*

- Finite affine parameter $\Sigma\Delta\tau_n < \infty$ — direct analog of finite affine parameter of a geodesic in [5].
- Inability to extend $J_O^+(C_N) = \emptyset$ — analog of inability to extend the geodesic in [5].

*Epistemic difference.* In [5] incompleteness is interpreted as ”absence of a point” in $M^4$ (singular point removed): extension of the geodesic leads outside $M^4$. In ODTOE incompleteness is interpreted as ”vanishing of the observer” on $\partial_B\mathcal{C}$: the point $C_N$ *exists* as a boundary object of $\overline{\mathcal{C}}$, but *carries no causal structure* ($J_O^+ = \emptyset$). This shifts the ontological accent: a singularity is not ”absence of spacetime” but ”absence of an observer” — conceptually consistent with the central axiom of ODTOE [13] §II.

### X.3. Substantive implication for closing C.T3

The sketch [18] §VII.4 in step 5 invokes: ”$\hat{O} = 0$ at $C_N$, hence $J_O^+(C_N) = \emptyset$ by definition of the causal structure [15] §III”. This step is marked in [18]: % [HYPOTHESIS: full formal proof requires Raychaudhuri analog in [13] §VI/§VII — see open status note below]. The present work closes the hypothesis:

- The Raychaudhuri-$\Phi$-analog is established (E.L1, lemma §VI.2).
- $\Phi$-iteration incompleteness gains explicit meaning via $J_O^+(C_N) = \emptyset$ (E.L4 + step 5 §IX.6).
- The connection with Geroch's geodesic incompleteness [5] is established structurally (§X.1–X.2).

This is the full closure of the sketch marker [18] §VII.4.

## XI. COMPARISON TO CLASSICAL HAWKING–PENROSE

### XI.1. Structural correspondence of hypotheses

The classical Hawking-Penrose 1970 theorem [6] (the unified version of Penrose 1965 [1] and Hawking 1966–67 I/II/III [2, 3, 4]) states: under (i) an energy condition, (ii) a generic convergence condition, (iii) a causality condition, (iv) the existence of a closed trapped surface (or an equivalent focusing-surface marker) — the spacetime is geodesically incomplete. Comparison with E.T1:

| Hawking-Penrose 1970 [6] | ODTOE E.T1 (present work) | Structural correspondence |
|---|---|---|
| Energy condition (WEC, NEC, or SEC) | ODTOE energy condition (E.F1) — lemma [18] §VII.1 | WEC direct analog; ODTOE makes WEC *derivable* from positivity of $B^2(1-\sigma)\Lambda$, not a postulate |

| Hawking-Penrose 1970 [6] | ODTOE E.T1 (present work) | Structural correspondence |
|---|---|---|
| Generic convergence condition | Standard focusing from (E.F11) + (E.F1) | Structural analog |
| Causality condition (no closed timelike curves) | Global hyperbolicity of $C_{\text{contr}}$ [18] §VI.2 + causal structure $J_O^+$ [15] §VI | Direct analog |
| Closed trapped surface $\mathcal{T}$ [1] | Trapped ODTOE-configuration $C_*$ (E.D1) | Structural analog via translation $\mathcal{T} \leftrightarrow C_*$, $J^+(\mathcal{T}) \leftrightarrow J_O^+(C_*)$ |
| Conclusion: geodesic incompleteness | Conclusion: $\Phi$-iteration incompleteness with $J_O^+(C_N) = \emptyset$ | Direct analog |

## XI.2. Differences and advantages of the ODTOE formulation

*Differences.*

- **Source of the energy condition.** In [6] WEC is taken as a *postulate* on the stress-energy tensor; in ODTOE WEC is *derived* from positivity of the B-functional and idempotence of the SYNC projector [17] L8.
- **Discreteness of $\Phi$-iteration.** In [6] focusing is analyzed on continuous geodesics; in ODTOE on the discrete $\Phi$-iteration sequence (with continuous limit). This gives a more explicit connection with the fundamental quantum nature of ODTOE.
- **Endpoint $C_N$ as a boundary object of $\overline{\mathcal{C}}$.** In [6] the singularity point is *absent* in $M^4$ (set $\overline{M} \setminus M$); in ODTOE the point $C_N$ *exists* in $\overline{\mathcal{C}}$ but carries no $J_O^+$-structure. This shifts the ontological accent from "deleted point" to "vanished observer".

*Structural advantages.*

- **Anti-circularity cleanliness.** In [6] WEC and the existence of a trapped surface are independent postulates; in ODTOE both are derived from the ODTOE formalism (WEC from L8 [17], trapped configuration from E.D1 + $J_O^+$ [15]).
- **Compatibility with the dynamic attractor.** The ODTOE formulation is explicitly compatible with the theory of attractors [20] §IV: the endpoint $C_N$ is a boundary object of the basin of the Fix$(\Phi)$ attractor, not a "remote singularity".

## XI.3. Position of E.T1 in the Senovilla 1998 taxonomy

By the taxonomy of [10] (Senovilla 1998 §3–§5) singularity theorems are classified by: (i) the energy condition type (WEC/NEC/SEC/DEC); (ii) the focusing-marker type (trapped surface, Cauchy surface, primordial focusing surface); (iii) the global structure type (global hyperbolicity, absence of closed timelike curves); (iv) the conclusion (geodesic incompleteness, curvature boundedness, extension breakdown).

*Position of E.T1.*

- Energy condition: **WEC** ( [10] §3, the weakest classical condition; sufficient for Penrose 1965 [1]).
- Focusing marker: **trapped configuration** ( [10] §4.2, Penrose-type).
- Global structure: **global hyperbolicity** ( [10] §4.1).
- Conclusion: **$\Phi$-iteration incompleteness**, analog of geodesic incompleteness.

By [10] §5 this is the Penrose-type subfamily ( [1]); E.T1 supplies the ODTOE instantiation in this subfamily. Comparator family: Hawking 1966 I [2], Hawking 1967 III [4] (Hawking-type, focusing surface); Hawking-Penrose 1970 [6] (unified). E.T1 does not cover the entire Hawking-Penrose 1970 family (extension to a focusing-surface of Hawking type is open question §XII), but covers the Penrose subfamily completely.

# XII. CONCLUSION AND OPEN QUESTIONS

## XII.1. Consolidated summary

The present work closes the marker *[OPEN: B-zero boundary topology]* of Article C [18] §VII.5 in the following sense:

1. The topological structure of the boundary $\partial_B \mathcal{C}$ is described via the trichotomy of Options A/B/C; Option A is ruled out; Options B and C are compatible, and for the purposes of proving E.T1 it suffices to use the structural property (SR) (E.F5) common to both surviving options (§III).
2. The criterion of finite affine parameter of $\Phi$-iteration is established by Theorem E.T2 (§IV.3) with explicit anti-circularity audit.
3. The formal definition of trapped ODTOE-configuration (E.D1) is given via $J^+_O$ with an explicit link to Penrose 1965 [1] (§V).
4. The ODTOE analog of the Raychaudhuri equation for $\Phi$-iteration is stated and proved (E.L1, §VI.2) with explicit anti-circularity audit.
5. The full ODTOE singularity theorem E.T1 (§VIII) is proved in five steps (§IX) with explicit anti-circularity audit (§IX.7).
6. The geodesic-incompleteness analog is discussed in the sense of Geroch 1968 [5] (§X).
7. The position of E.T1 in the Senovilla 1998 taxonomy [10] is established (§XI.3).

In the corpus numbering C.T3 [18] §VII.3 is promoted from status: HYPOTHESIS to status: THEOREM *logically* via E.T1.

## XII.2. Open questions and prospects

**O1. Physical removal of the marker C.T3 (status: HYPOTHESIS) in [18].** The present work closes the marker *logically* (via E.T1), but *physically* the file [18] remains in its current state. Removal of the marker [18] (7.3) and update of C.T3 from status: HYPOTHESIS to status: THEOREM is a separate task (RT-1.5 ROADMAP, AC-8). It is not part of the commit window of the present paper (BL-24).

**O2. Final choice of Option B vs. Option C in the trichotomy §III.4.** The marker *[OPEN: option selection]* remains open. Resolution requires analysis of the conformal structure of the observation operator $\hat{O}$ (paper "Conformal structure of $\hat{O}$ in ODTOE" — future work of the corpus).

**O3. Generalization of E.T1 to the Hawking-Penrose 1970 [6] family.** Coverage of the Penrose subfamily is complete; extension to a focusing surface of Hawking 1966–67 I/II/III [2, 3, 4] type (a focusing 3-surface instead of a trapped 2-surface) is an open task. Technically it requires an analog of the focusing operator for timelike congruences in ODTOE.

**O4. Global structure of $\overline{\mathcal{C}}$ as a manifold with corners.** Option C of the trichotomy points to a stratified structure of $\partial_B\mathcal{C}$ with corners and edges; formalization in the spirit of Lee [12] Ch. 16 (manifolds with corners) is an open task.

**O5. Numerical verification of $\Phi$-iteration in the neighbourhood of $\partial_B\mathcal{C}$.** Empirical confirmation of $\Phi$-iteration trajectories with $\Sigma\Delta\tau_n < \infty$ via numerical modelling of (E.F2) in the collapse regime is an open task (requires adaptation of the framework [20] §IV.3 to the $\partial_B$-zone).

## XII.3. Connection with the ODTOE programme

In the programme of *[18]* §XIV.3 stage 3 of closing the three-stage programme was presented in [18] as "Einstein equation as $\Phi$-self-consistency + dual-path Bianchi + ODTOE singularity theorem analog". Of these three components the first two (C.T1, C.T2) are fully proved in [18]; the third (C.T3) is presented in [18] as a sketch with explicit HYPOTHESIS marker.

The present work closes the third component:

- Stage 1 (tensor layer): closed by [16] (Article A).
- Stage 2 (source): closed by [17] (Article B).
- Stage 3 (closure): closed by [18] for C.T1 and C.T2; for C.T3 — closed by the present work (Article E).

This is the *last* required component for full closure of the programme [18] §XIV.3 in the sense of an ODTOE analog of the classical singularity theory. From the corpus standpoint this synchronizes the ODTOE gravitational stack with the classical Hawking-Penrose taxonomy [10] at the theorem level.

*Position within the T0 programme and the explicit closure delegation.* The full synthesis of the ODTOE gravitational programme is encapsulated in paper [19]

(`ODTOE_einstein_full_closure`): it unifies papers [16] (tensor layer A), [17] (source B), [18] (closure C) and [15] (causal structure D) into a single closure T0 $\to$ A $\to$ B $\to$ C $\to$ XL of the programme and explicitly delegates the proof of C.T3 [18] §VII.5 (correspondingly, closing the marker *[OPEN: B-zero boundary topology]*) to a *separate paper* of the series. The present Article E is precisely that delegated work: it closes the remaining open component in [19], promotes C.T3 to status THEOREM logically, and thereby turns the synthesis [19] from "a programme with one open marker" into a full closure of the gravitational chain. After the present work every statement on which [19] §VIII of the closure rests has status THEOREM in the corpus; the only residual step is physical removal of the marker in file [18] (open question O1 §XII.2).


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## CONFLICT OF INTEREST

The author declares no conflict of interest concerning the content of the present work.

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*Note on order.* The reference list is organized into three conceptual blocks [L-35-ext]: (1) foundational classical works on singularity theorems (Penrose 1965; Hawking 1966–67 I/II/III; Geroch 1968; Hawking-Penrose 1970), monographs (Hawking-Ellis 1973; Penrose 1979 in Einstein Centenary Survey; Wald 1984), review (Senovilla

1998), and general topology / smooth manifolds (Munkres 2000; Lee 2012); (2) author preprints on the ODTOE corpus in the order of first citation in the text. A reference-data section is absent, since the present paper is a purely topological work closing C.T3 §VII.5 [OPEN] of [18].


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