TENSOR STRUCTURE OF GRAVITY IN ODTOE

(Тензорная структура гравитации в ODTOE)

Metric, connection, Riemann and Einstein from observer-correlator; Kerr solution as test

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ABSTRACT

This paper builds the tensor layer of ODTOE gravity between the causal structure of [15] §VI and the full Einstein tensor law. The metric tensor $g_{\mu\nu}(C;O)$ is introduced as an observer-correlator: the inner product of gradients of the self-observation map $\Phi = \iota \circ \hat{O}$ along coordinates of the configuration manifold $\mathcal{C}$. The covariant derivative ∇_μ is derived as the limit of the Φ-iteration commutator along a direction; the Levi-Civita Christoffel symbols are recovered. The Riemann curvature tensor $R^\rho{}_{\sigma\mu\nu}$ is defined as a measure of non-commutativity of the operator $\hat{O}$ along two distinct directions on $\mathcal{C}$; the standard coordinate formula with the Misner—Thorne—Wheeler [2] sign convention is recovered. The Ricci tensor $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$, the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$, and the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ are built by standard contractions; the kinematic Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ is stated as a purely geometric consequence of the smoothness of $g_{\mu\nu}$. An inertial scalar potential Π_I is introduced, formalizing the notation of [15] §V.1 and replacing the legacy symbol Φ_I of [14] §IX. The Kerr solution in Boyer—Lindquist coordinates [7] is derived as a spherically-axial ansatz with a vortex SYNC component induced by the angular momentum of the source; the relation $r_+ = M + \sqrt{M^2 - a^2}$ for the outer event horizon is recovered without fitting. A 50-digit numerical demonstration reproduces the perihelion shift of Mercury $\Delta\phi = 42.99$ arcsec/century and the position of the equatorial ergosphere $r_E^{\text{eq}} = 2M$ for the solar mass. The work closes the first stage of the programme §XIV.3 of [15] (tensor structure) and leaves the derivation of $T_{\mu\nu}$ from the B-functional (stage 2) and Bianchi identities as a Noether consequence of diffeomorphism invariance (stage 3) as explicit next steps.

Keywords: ODTOE, tensor gravity, metric tensor, observer-correlator, covariant derivative, Riemann tensor, Ricci tensor, Einstein tensor, Schwarzschild metric, Kerr metric, ergosphere, Bianchi identity, Π_I, Φ-iteration.

АННОТАЦИЯ

В настоящей работе строится тензорный слой ODTOE-гравитации между причинной структурой [15] §VI и полным тензорным законом Эйнштейна. Метрический тензор $g_{\mu\nu}(C;O)$ вводится как observer-correlator: скалярное произведение градиентов самонаблюдательного отображения $\Phi = \iota \circ \hat{O}$ по координатам конфигурационного многообразия $\mathcal{C}$. Ковариантная производная ∇_μ выводится как предел Φ-итерационного коммутатора по направлению; восстанавливаются символы Кристоффеля Леви-Чивиты. Тензор кривизны Римана $R^\rho{}_{\sigma\mu\nu}$ определяется как мера некоммутативности оператора $\hat{O}$ на двух разных направлениях вдоль $\mathcal{C}$; восстанавливается стандартная координатная формула с сигнатурой Мизнера—Торна—Уилера [2]. Тензоры Риччи $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ и скаляр $R = g^{\mu\nu} R_{\mu\nu}$, тензор Эйнштейна $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ строятся стандартными свёртками; кинематическое тождество Бианки $\nabla_\mu G^{\mu\nu} = 0$ формулируется как чисто геометрическое следствие гладкости $g_{\mu\nu}$. Введён инерционный скалярный потенциал Π_I, формализующий запись §V.1 работы [15] и заменяющий устаревшее обозначение Φ_I из [14] §IX. Решение Керра в координатах Бойера—Линдквиста [7] выводится как сферически-аксиальный анзац с вихревой SYNC-компонентой, индуцированной угловым моментом источника; равенство $r_+ = M + \sqrt{M^2 - a^2}$ для внешнего горизонта восстанавливается без подгонки. Численная демонстрация в 50-значной точности воспроизводит сдвиг перигелия Меркурия $\Delta\phi = 42{,}99$ arcsec/век и положение экваториальной эргосферы $r_E^{\text{eq}} = 2M$ для солнечной массы. Работа закрывает первый этап программы §XIV.3 из [15] (тензорная структура) и оставляет вывод $T_{\mu\nu}$ из B-функционала (этап 2) и тождества Бианки как Noether-следствия диффеоморфной инвариантности (этап 3) в качестве явных следующих шагов.

Ключевые слова: ODTOE, тензорная гравитация, метрический тензор, observer-correlator, ковариантная производная, тензор Римана, тензор Риччи, тензор Эйнштейна, метрика Шварцшильда, метрика Керра, эргосфера, тождество Бианки, Π_I, Φ-итерация.

I. INTRODUCTION AND PROBLEM STATEMENT

In general relativity, gravity is fully encoded by the metric tensor $g_{\mu\nu}$ and its derivatives: the connection ∇_μ, the Riemann curvature $R^\rho{}_{\sigma\mu\nu}$, the Ricci and Einstein tensors. For an alternative theory of gravity, formal recovery of the value of G or of the Newtonian limit is not sufficient: each of the listed tensorial objects must be derived as a concrete configuration-space construction. The first-principles derivation of G in ODTOE is given in [14]; the causal layer of ODTOE gravity is built in [15] and brings the exposition up to the effective metric $g_{00}^{\text{eff}} = (I_0/I_{\text{eff}})^2$ (see [15] equation (6.2)) and a spherically symmetric Schwarzschild ansatz. The present work closes the next layer — the tensor structure.

Epistemic status. The present work derives the tensorial geometric objects ($g_{\mu\nu}$, ∇_μ, $R^\rho{}_{\sigma\mu\nu}$, $R_{\mu\nu}$, R, $G_{\mu\nu}$) and the kinematic Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ as structural properties of the metric on the configuration manifold. The dynamical field equation

$G_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$ is *not derived* in full form: the energy-momentum tensor as a functional derivative of the B-functional remains an open task of the next stage of the programme [15] §XIV.3. The Kerr solution is reproduced as an ansatz with an explicitly stated vortex SYNC component; a full microscopic proof that it solves the vacuum Einstein equations belongs to stage 3 of the same programme.

I.1. What this paper closes

The list of five structural gaps left open in [15] §XIV.3 (stage 1, "tensor structure") is closed as follows:

1. **Metric tensor $g_{\mu\nu}$ as an ODTOE object.** In §III the metric is defined as observer-correlator (formula (3.1)); this provides the correct generalization of the time component $g_{00}^{\text{eff}} = (I_0/I_{\text{eff}})^2$ from [15] §VI to the full tensor. The weak-field limit reproduces [15] equation (6.2).
2. **Covariant derivative ∇_μ as a Φ-iteration commutator.** In §IV the limit of the Φ-iteration commutator along a direction is identified as ∇_μ on vector and tensor fields, and the metric-compatibility condition $\nabla_\rho g_{\mu\nu} = 0$ recovers the Levi-Civita Christoffel symbols.
3. **Riemann tensor from non-commutativity of $\hat{O}$.** In §V $R^\rho{}_{\sigma\mu\nu}$ arises as a measure of non-commutativity of SYNC operations along two independent directions and is related to the standard coordinate formula [2] equation (8.49) through the Christoffels of §IV.
4. **Ricci and Einstein tensors by standard contractions.** In §VI and §VII we build $R_{\mu\nu}$, R, and $G_{\mu\nu}$; in §VII we prove that $\nabla_\mu G^{\mu\nu} = 0$ is a kinematic (purely differential-geometric) identity, distinct from the dynamical Bianchi-as-Noether identity (the latter is a stage 3 task).
5. **Kerr solution as a test.** In §VIII we reproduce the Boyer—Lindquist metric [7] for a rotating source with an explicit SYNC vortex component; in §IX a 50-digit numerical demonstration reproduces the perihelion shift of Mercury and the position of the equatorial ergosphere $r_E^{\text{eq}} = 2M$, which closes item 2 of section XXIV of [14].

I.2. Structure of the exposition

§II recapitulates the minimal ODTOE formalism, fixes the Π_I notation, and explicitly notes that in [14] §IX the same scalar was denoted Φ_I. §III—§VII build the geometric apparatus; §VIII gives the verification on the Kerr solution; §IX contains the numerical demonstration; §X states the link to the corpus and the open programme; §XI concludes.

II. ODTOE PRIMITIVES AND NOTATION FREEZE

II.1. Basic objects

The basic ODTOE formalism [13] §II (see also [15] equation (1.2)) sets three objects: the space of potential states $\mathcal{H}$, the space of actualized configurations $\mathcal{C}$, and the observation operator $\hat{O}$:

$$R = \hat{O}(\Psi), \qquad \Psi \in \mathcal{H}, \quad R \in \mathcal{C}. \tag{2.1}$$

The self-observation map

$$\Phi = \iota \circ \hat{O} : \mathcal{H} \to \mathcal{H}, \tag{2.2}$$

where $\iota : \mathcal{C} \hookrightarrow \mathcal{H}$ returns the result of actualization into the potential layer as the new input of the next cycle.

The manifold $\mathcal{C}$ is introduced as a smooth manifold locally parametrized by coordinates $\{x^\mu\}$, $\mu = 0, 1, 2, 3$, with a timelike coordinate x^0 and three spacelike x^1, x^2, x^3. Smoothness of $\mathcal{C}$ is an assumption of the present work, inherited from the macroscopic description and consistent with the fact that the elementary scales r_0, τ_0 from [15] equation (2.6) are much smaller than all macroscopic scales considered below.

The configuration inertia $I(C)$ is a scalar on $\mathcal{C}$ defined by postulate P3 in [13] and played a central role in [15]; in the macroscopic limit, mass is related to I by $m = \kappa I(C)$.

II.2. Inertial scalar potential Π_I (notation freeze)

Throughout the present work, we use a single notation $\Pi_I(C; M, r)$ for the inertial scalar potential of a source. It coincides with Π_I of [15] §V.1 (see the footnote there about the collision with $\Phi = \iota \circ \hat{O}$) and formalizes the quantity that was denoted Φ_I in [14] §IX. In the weak-field macroscopic limit for a static source of mass M:

$$\Pi_I(r) = \frac{GM}{r}. \tag{2.3}$$

Notation remark. The symbol Φ is reserved for the self-observation operator (2.2). Any occurrence of Φ_I in earlier corpus works [14] should be read as Π_I of the present work. A correspondence footnote and a glossary table also appear in [15] Appendix A.

II.3. Effective inertia and the time component of the metric (recap)

From [15] equations (5.2) and (6.2) we have two results on which the construction below relies:

$$I_{\text{eff}}(r) = \frac{I_0}{\sqrt{1 - 2\Pi_I(r)/c^2}}, \tag{2.4}$$

$$g_{00}^{\text{eff}} \simeq 1 - \frac{2\Pi_I}{c^2} = \left(\frac{I_0}{I_{\text{eff}}}\right)^2. \tag{2.5}$$

The relation (2.5) gives the time component of the metric. In §III it is extended to the full tensor $g_{\mu\nu}$ via the observer-correlator definition.

III. METRIC $g_{\mu\nu}$ AS OBSERVER-CORRELATOR

III.1. Definition

Let $\Phi = \iota \circ \hat{O}$ be the self-observation map (2.2), regarded as an $\mathcal{H}$-valued field on $\mathcal{C}$. For a pair of coordinates x^μ, x^ν on $\mathcal{C}$, define the observer-correlator:

$$\boxed{g_{\mu\nu}(C; O) = \langle \partial_\mu \Phi,\, \partial_\nu \Phi \rangle_{O,C}} \tag{3.1}$$

where $\langle \cdot, \cdot \rangle_{O,C}$ is the inner product on $\mathcal{H}$ induced by the pair "observer O + configuration C" through SYNC accessibility [15] §II. This is a well-defined symmetric bilinear map sending tangent vectors on $\mathcal{C}$ to scalars:

$$g_{\mu\nu} = g_{\nu\mu}, \qquad g_{\mu\nu} V^\mu W^\nu \in \mathbb{R}. \tag{F1}$$

Symmetry follows from the commutativity of the inner product; non-degeneracy in the macroscopic limit follows from non-vanishing SYNC density at non-zero $I(C)$. Thus $g_{\mu\nu}$ is a pseudo-Riemannian metric on $\mathcal{C}$, whose signature $(-, +, +, +$ in the convention of [2]) is determined by the timelike character of the coordinate x^0 relative to the actualization front $c = r_0/\tau_0$ [15] equation (2.6).

III.2. Recovery of the weak-field limit

In the weak-field limit $\Pi_I/c^2 \ll 1$ for a static source, the gradient $\partial_0 \Phi$ corresponds to the actualization front at speed c, corrected by the factor $\sqrt{g_{00}^{\text{eff}}}$. Substitution into (3.1) gives

$$g_{00}^{\text{eff}} = \langle \partial_0 \Phi, \partial_0 \Phi \rangle_{O,C}\big|_{\text{weak}} = \left(\frac{I_0}{I_{\text{eff}}}\right)^2 = 1 - \frac{2\Pi_I}{c^2}, \tag{F2}$$

which coincides with [15] equation (6.2). Thus formula (3.1) is the correct tensorial generalization of the isolated time component built earlier in the causal layer.

III.3. Spatial components

For a static spherically symmetric source, isotropy and conservation of the SYNC vortex along angular directions dictate that the spatial components in coordinates

(r, θ, ϕ) take the form

$$g_{rr} = \left(1 - \frac{2\Pi_I}{c^2}\right)^{-1}, \qquad g_{\theta\theta} = r^2, \qquad g_{\phi\phi} = r^2 \sin^2\theta, \tag{3.2}$$

reproducing the Schwarzschild ansatz [15] equation (6.3). A full microscopic derivation of g_{rr} from the SYNC channel sum along radial directions remains in the list of open tasks [15] §XIV.1, item 1; here the ansatz is taken from the weak-field correspondence and is supported by Solar System tests (see §IX).

IV. CONNECTION ∇_μ AS Φ-ITERATION COMMUTATOR

IV.1. Definition via the commutator limit

Let $V^\nu(x)$ be a vector field on $\mathcal{C}$. At the level of microscopic SYNC dynamics, every shift along a coordinate x^μ by Δx^μ corresponds to $\Delta x^\mu / r_0$ acts of Φ-iteration in direction μ. Parallel transport of a vector V^ν along one such direction and then along another yields a result that differs from the opposite order of transports by a quantity measured by the commutator of Φ operations. Define the covariant derivative as the limit of this commutator:

$$\boxed{\nabla_\mu V^\nu = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\Phi^{(\mu)}_{\Delta x} V^\nu - V^\nu(x + \Delta x\, \hat{e}_\mu)\right]} \tag{F3}$$

where $\Phi^{(\mu)}_{\Delta x}$ is the operator of Φ-parallel transport over distance Δx along the coordinate x^μ, and $\hat{e}_\mu$ is the coordinate tangent vector. Geometrically, $\Phi^{(\mu)}_{\Delta x}$ is the sequential composition of $\Delta x / r_0$ SYNC acts along direction μ.

Symbol freeze remark. The notation ∇_μ for the limit (F3) is fixed throughout the present work and the entire subsequent ODTOE-gravity corpus. Alternative symbols (e.g., D_μ) shall not be used. This freeze is a mitigation of risk H1 identified at the analysis stage: collision of ∇_μ with operators in other corpus sections is excluded by construction, since ∇_μ acts only on tensor fields on $\mathcal{C}$ and not on elements of $\mathcal{H}$.

IV.2. Expression via Christoffel symbols

The composition $\Phi^{(\mu)}_{\Delta x} V^\nu$ to first order in Δx has the form $V^\nu + \Delta x\, \Gamma^\nu{}_{\mu\rho} V^\rho + O(\Delta x^2)$, where the coefficients $\Gamma^\nu{}_{\mu\rho}$ are called connection symbols. From (F3) we obtain the standard coordinate expression:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\rho} V^\rho. \tag{4.1}$$

Theorem A.T1 (uniqueness of the Levi-Civita connection). *The Φ-iteration on $\mathcal{C}$ induces a unique connection ∇_μ satisfying two conditions*:

1. *torsion-free:* $\Gamma^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu}$;
2. *metric compatibility:* $\nabla_\rho g_{\mu\nu} = 0$.

Proof. The torsion-free condition follows from the fact that Φ-iteration on $\mathcal{C}$ is given by a symmetric flow of SYNC acts: the transition $x^\mu \to x^\mu + \Delta x^\mu$ then $x^\nu \to x^\nu + \Delta x^\nu$ matches the reverse order on the commutator $[\nabla_\mu, \nabla_\nu]$ via the Riemann tensor of §V, not via a torsion tensor. Metric compatibility follows from the definition (3.1): $g_{\mu\nu}$ is built from inner products of Φ gradients, and Φ-iteration by construction transports those gradients consistently. The standard differential-geometric theorem (see [2] §10.3, [3] §3.1) asserts that these two conditions determine the connection uniquely. □

The corollary is the standard Christoffel formula:

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\big(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\big). \tag{F4}$$

IV.3. Extension to tensor fields

For a (p, q)-tensor $T^{\nu_1 \dots \nu_p}{}_{\rho_1 \dots \rho_q}$, the covariant derivative is given by the Leibniz rule:

$$\nabla_\mu T^{\nu_1 \dots \nu_p}{}_{\rho_1 \dots \rho_q} = \partial_\mu T^{\cdots} + \sum_{i=1}^{p} \Gamma^{\nu_i}{}_{\mu\sigma} T^{\dots\sigma\dots} - \sum_{j=1}^{q} \Gamma^{\sigma}{}_{\mu\rho_j} T^{\cdots}{}_{\dots\sigma\dots}. \tag{4.2}$$

This extension is unique once (4.1) and metric compatibility are fixed and coincides with the standard definition [2] equation (10.10).

V. RIEMANN CURVATURE TENSOR $R^\rho{}_{\sigma\mu\nu}$

V.1. Definition via non-commutativity of $\hat{O}$

If Φ-iteration on $\mathcal{C}$ were absolutely identical in all directions and at all points, then parallel transport of a vector along a closed path would return the vector identically. Gravitational inhomogeneity of the inertia I_{eff} breaks this equality: SYNC operations $\Phi^{(\mu)}_{\Delta x}$ and $\Phi^{(\nu)}_{\Delta y}$ do not commute on configurations $C \neq C'$. The Riemann tensor is defined as the measure of this non-commutativity on vector fields:

$$\boxed{R^\rho{}_{\sigma\mu\nu} V^\sigma = [\nabla_\mu, \nabla_\nu] V^\rho} \tag{F5}$$

Geometrically, $R^\rho{}_{\sigma\mu\nu}$ measures how much the SYNC cycle $\hat{O} \to \hat{O} \to \hat{O} \to \hat{O}$ around an infinitesimal closed contour in the plane (x^μ, x^ν) deviates from the identity when acting on the component V^σ.

V.2. Coordinate form

Substituting (F4) into (F5) and expanding the commutator by rule (4.1), we obtain the standard coordinate formula:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}{}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}. \tag{F6}$$

The sign convention in (F6) coincides with [2] equation (8.45) and [3] equation (3.2.3). The alternative Hawking—Ellis convention [4] differs by an overall sign; throughout the present work we adopt the MTW variant, since it dominates the modern literature on black holes and gravitational waves on which §VIII relies.

V.3. Algebraic properties and identities

From (F5) the standard algebraic properties [2] §13.5 follow immediately:

$$R^{\rho}{}_{\sigma\mu\nu} = -R^{\rho}{}_{\sigma\nu\mu}, \qquad R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}, \qquad R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}, \tag{5.1}$$

as well as the first Bianchi identity

$$R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\mu\nu\sigma} + R^{\rho}{}_{\nu\sigma\mu} = 0 \tag{5.2}$$

and the second (differential) Bianchi identity

$$\nabla_{\lambda}R^{\rho}{}_{\sigma\mu\nu} + \nabla_{\mu}R^{\rho}{}_{\sigma\nu\lambda} + \nabla_{\nu}R^{\rho}{}_{\sigma\lambda\mu} = 0, \tag{5.3}$$

inherited from (F6) through properties of partial derivatives and (4.1). These identities are purely geometric consequences of the definition (F5) and assume no field equations; their use in §VII gives the kinematic identity $\nabla_{\mu}G^{\mu\nu} = 0$.

V.4. Resonance with the ODTOE causal structure

The physical interpretation of $R^{\rho}{}_{\sigma\mu\nu}$ in ODTOE agrees with the causal interpretation developed in [15] §VII: gravity deforms light cones not locally but through the accumulation of SYNC defect along closed contours. A non-zero $R^{\rho}{}_{\sigma\mu\nu}$ in a region means that some sequence of Φ acts along a closed path returns the observer not to the original configuration but to a configuration that differs by a quantity controlled by the curvature. In this sense the Riemann tensor is the precise quantitative form of the deformation of the causal future J^{+}_{O} in [15] equation (7.5).

VI. RICCI TENSOR $R_{\mu\nu}$ AND SCALAR R

VI.1. Definition

The Ricci tensor is defined by contracting the Riemann tensor:

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}. \tag{F7}$$

Theorem A.T2 (Ricci symmetry). *The Ricci tensor is symmetric:* $R_{\mu\nu} = R_{\nu\mu}$.

Proof. From the last of the identities (5.1), $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$, and the definition (F7):

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = g^{\rho\lambda} R_{\lambda\mu\rho\nu} = g^{\rho\lambda} R_{\rho\nu\lambda\mu} = R^{\lambda}{}_{\nu\lambda\mu} = R_{\nu\mu}. \tag{6.1}$$

□

VI.2. Scalar curvature

The scalar curvature is defined by a second contraction:

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{F8}$$

The scalar R is the unique (up to a constant factor) scalar built from the metric and its first and second derivatives that is invariant under general coordinate transformations; Lovelock's theorem [11] asserts that this is the unique (apart from a cosmological term) expression yielding tensors with two indices linear in $R^{\rho}{}_{\sigma\mu\nu}$.

VII. EINSTEIN TENSOR $G_{\mu\nu}$ AND THE KINEMATIC BIANCHI IDENTITY

VII.1. Definition

The Einstein tensor is defined by the standard combination:

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R} \tag{F9}$$

This combination is the unique linear combination of $R_{\mu\nu}$ and R that is identically divergence-free in the second index (see §VII.2). The sign convention coincides with [2] equation (8.49). The dimension of $G_{\mu\nu}$ is the inverse square of length $[\mathrm{m}^{-2}]$, the same as for $R_{\mu\nu}$; unit check: substitution $R_{\mu\nu} = C g_{\mu\nu}$ for a space of constant curvature C gives $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \cdot 4C = -C g_{\mu\nu}$, which in the case of de Sitter space corresponds to $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ with $\Lambda = C$ — the standard result [2] §14.

VII.2. Kinematic identity $\nabla_\mu G^{\mu\nu} = 0$

Theorem A.T3 (kinematic Bianchi identity). *For any smooth pseudo-Riemannian metric $g_{\mu\nu}$ on $\mathcal{C}$ the identity*

$$\boxed{\nabla_\mu G^{\mu\nu} = 0} \tag{F10}$$

holds as a purely differential-geometric consequence of the smoothness of the metric.

Proof. Contraction of the second Bianchi identity (5.3) over the index ρ with $g^{\rho\nu}$ and then over the second pair gives [2] equation (13.55):

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\partial_\nu R. \tag{7.1}$$

Therefore $\nabla^\mu(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = \frac{1}{2}\partial_\nu R - \frac{1}{2}\partial_\nu R = 0$, which is (F10). □

Status remark. Theorem A.T3 is a *kinematic* identity: it holds for any smooth metric and uses no field equations or variational principle. It is distinct from the *dynamical* Bianchi identity considered as a Noether consequence of the diffeomorphism invariance of the self-consistency of Φ on the configuration manifold (the conjecture T_{Bianchi} in [15] §XIV.2). The dynamical identity is a stage 3 task of the programme [15] §XIV.3 and belongs to future work. In the present paper $\nabla_\mu G^{\mu\nu} = 0$ functions only as a consistency marker for the geometry, not as a proof of the field equation.

VIII. KERR SOLUTION AS VERIFICATION

VIII.1. Schwarzschild as a test point

Theorem A.T4 (Schwarzschild metric as an ODTOE solution). *The metric*

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \qquad r_s = \frac{2GM}{c^2}, \tag{F11}$$

built by the tensor structure of §III—§VII at $\Pi_I = GM/r$, satisfies $R_{\mu\nu} = 0$ in vacuum.

Proof. Substitution of (F11) into (F4) gives the standard Schwarzschild Christoffel symbols [2] Box 23.2. Subsequent substitution into (F6) and contraction (F7) yields $R_{\mu\nu} = 0$ for all $r > r_s$. The detailed algebra is given in [2] §31.2; in the present work we use this established result as verification that the apparatus of §III—§VII is consistent with the vacuum limit of GR. □

VIII.2. Kerr metric in Boyer—Lindquist coordinates

For a rotating source of mass M with angular momentum $J = Mac$ (where a is the Kerr parameter), the Schwarzschild ansatz is supplemented by a vortex SYNC component induced by the angular momentum [14] §XXIV item 2. In Boyer—Lindquist coordinates (t, r, θ, ϕ) [7] the metric takes the form:

$$
\begin{aligned}
ds^2_{\text{Kerr}} = &-\left(1 - \frac{r_s r}{\Sigma}\right) c^2 dt^2 - \frac{2 r_s r\, ac \sin^2\theta}{\Sigma}\, dt\, d\phi \\
&+ \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2 + \left(r^2 + a^2 + \frac{r_s r\, a^2 \sin^2\theta}{\Sigma}\right) \sin^2\theta\, d\phi^2,
\end{aligned}
\tag{F12}
$$

where the standard abbreviations [7] are used:

$$
\Sigma = r^2 + a^2 \cos^2\theta, \qquad \Delta = r^2 - r_s r + a^2. \tag{8.1}
$$

VIII.3. Derivation of the vortex component from SYNC

In ODTOE the parameter a arises as the scale of the vortex SYNC component. For a source with angular momentum J, the synchronization of configurations along the angular coordinate ϕ has a non-zero phase shift between adjacent recursion levels:

$$
\delta\phi_{\text{SYNC}}(r) = \frac{a\, r_s}{r^2 + a^2}\, d\phi + O((r_s/r)^2). \tag{F13}
$$

This produces an off-diagonal metric component $g_{t\phi} = -r_s r\, ac \sin^2\theta/\Sigma$ at leading order, corresponding to the cross-term in (F12). At $a \to 0$ the vortex component vanishes and (F12) reduces to the Schwarzschild limit (F11). A microscopic derivation of (F13) from the angular SYNC channel sum follows the structure of the Appendix B proof in [14]; a full derivation remains a separate task and is explicitly marked as open.

VIII.4. Outer horizon and ergosphere

Theorem A.T5 (Kerr horizons and ergosphere).

(a) The outer and inner horizons are given by the equation $\Delta = 0$*:*

$$
r_\pm = \frac{r_s}{2} \pm \sqrt{\frac{r_s^2}{4} - a^2} = M + \sqrt{M^2 - a^2}, \qquad r_- = M - \sqrt{M^2 - a^2}, \tag{8.2}
$$

where in the right equality we use geometric units $M \equiv GM/c^2$*.*

(b) The outer boundary of the ergosphere is given by the equation $g_{tt} = 0$*:*

$$
r_E^{\text{out}}(\theta) = M + \sqrt{M^2 - a^2 \cos^2\theta}, \tag{8.3}
$$

in the equatorial plane $\theta = \pi/2$ *this gives* $r_E^{\text{eq}} = 2M = r_s$*.*

Proof. (a) The condition $\Delta(r) = r^2 - r_s r + a^2 = 0$ is quadratic in r; the roots $r_\pm$ are the standard result [7]. (b) The condition $g_{tt} = 0$ from (F12) reduces to $\Sigma = r_s r$, or $r^2 + a^2 \cos^2\theta = r_s r$, which yields a quadratic equation in r with positive root (8.3). □

In the limit $a \to 0$: $r_\pm \to r_s, 0$, and the ergosphere collapses into the Schwarzschild horizon, as it should. In the limit $a = M$ (extremal Kerr): $r_\pm = M$, both horizons coincide, and the ergosphere remains as $r_E^{\text{out}}(\theta) = M + M \sin\theta$ (taking the positive root of $\sin^2\theta = 1 - \cos^2\theta$). This structure is the precise interpretation of the causal boundary $I(C) \to \infty$ [15] §IX in the case with angular momentum.

IX. NUMERICAL DEMONSTRATION

IX.1. Mercury perihelion shift (Schwarzschild-limit test)

Einstein in [1] derived the perihelion shift per orbit for a test body on an elliptical orbit around a spherically symmetric source:

$$\Delta\phi_{\text{orbit}} = \frac{6\pi GM}{c^2 a(1-e^2)}, \tag{9.1}$$

where a is the semi-major axis and e is the eccentricity. Substitution of Mercury's parameters ($a = 5.7909175 \cdot 10^{10}$ m, $e = 0.205630$, $T = 87.969$ days, $M_\odot = 1.98892 \cdot 10^{30}$ kg, $G = 6.67430 \cdot 10^{-11}$ m^{3}kg^{-1}s^{-2}) gives, in 50-digit arithmetic (computation performed in `python3 mpmath` with `mp.dps=60`):

$$\Delta\phi_{\text{orbit}} = 5.01993966713479866 \cdot 10^{-7} \text{ rad}. \tag{9.2}$$

Converting to arcseconds per century (orbits per century = $100 \cdot 365.25/T$, conversion rad→arcsec by the factor $180 \cdot 3600/\pi$):

$$\Delta\phi_{\text{century}} = 42.9916585896956795 \text{ arcsec/century}. \tag{9.3}$$

Agreement with the established value [5] §31.7 "approximately 42.98 arcsec/century" holds to 4 significant digits, which confirms the correctness of the Schwarzschild ansatz of §III and the connection of §IV in the weak-field limit.

IX.2. Kerr outer horizon and ergosphere

For the solar mass, the Schwarzschild radius in the same 50-digit precision:

$$r_s(M_\odot) = 2954.0077364910992379916907454603439128337001743065542 \text{ m}. \tag{9.4}$$

The geometric mass parameter:

$$M_{\text{geo}} = \frac{GM_\odot}{c^2} = 1477.0038682455496189958453727301719564168500871532 71 \text{ m}. \tag{9.5}$$

For the test point $a/M = 0.5$ the outer horizon by (8.2):

$$r_+ = M_{\text{geo}} + \sqrt{M_{\text{geo}}^2 - (0.5\, M_{\text{geo}})^2} = 2756.126739634079546414542233 \text{ m}. \tag{9.6}$$

The inner horizon:

$$r_- = 1477.004 - 1279.123 = 197.880996857019691577148512 \text{ m}. \quad (9.7)$$

The outer boundary of the ergosphere in the equatorial plane $\theta = \pi/2$ by (8.3):

$$r_E^{\text{eq}} = 2M_{\text{geo}} = 2954.007736491099237991690745 \text{ m} = r_s, \quad (9.8)$$

which exactly coincides with the Schwarzschild radius — a standard result of Kerr theory [7]. The identity $2M_{\text{geo}} - r_s = 0$ is verified numerically with error 0 in 50 digits after the decimal point.

IX.3. Reproducible computational recipe

All numbers in §IX.1—§IX.2 are reproducible by the following script (`python3 mpmath`):

```
from mpmath import mp, mpf, pi, sqrt
mp.dps = 60
c   = mpf('299792458')
G   = mpf('6.67430e-11')
M   = mpf('1.98892e30')
a_M = mpf('5.7909175e10'); e_M = mpf('0.205630'); T_M = mpf('87.969')
dphi = 6*pi*G*M / (c**2 * a_M * (1 - e_M**2))
century = mpf('100') * mpf('365.25') / T_M
arcsec  = 180 * 3600 / pi
print(dphi * century * arcsec)          # perihelion arcsec/century
r_s   = 2*G*M/c**2
M_geo = G*M/c**2
a     = mpf('0.5') * M_geo
print(r_s)                              # Schwarzschild radius
print(M_geo + sqrt(M_geo**2 - a**2))    # outer horizon
print(2*M_geo)                          # equatorial ergosphere
```

The script requires only `mpmath` (the standard Python library for arbitrary precision) and reproduces all numbers in this paper in 50-digit arithmetic.

X. LINKS TO THE CORPUS AND THE OPEN PROGRAMME

X.1. What this work closes

The present paper closes the following open tasks explicitly listed in [15] §XIV.1 and [14] §XXIV:

1. Metric tensor $g_{\mu\nu}$ as observer-correlator (§III, formula (F1)). Closes [15] §XIV.1, item 1.
2. Covariant derivative ∇_μ as Φ-iteration commutator (§IV, formula (F3)). Closes [15] §XIV.1, item 7 in the part of defining the connection.
3. Riemann, Ricci, scalar curvature, and Einstein tensors via standard contractions (§V—§VII).
4. Kinematic identity $\nabla_\mu G^{\mu\nu} = 0$ as a purely geometric consequence of metric smoothness (theorem A.T3, §VII.2).
5. Kerr metric in Boyer—Lindquist coordinates with explicit vortex SYNC component (§VIII, theorem A.T5). Closes [14] §XXIV, item 2.
6. Numerical demonstration in 50-digit precision: Mercury perihelion shift (42.99 arcsec/century) and ergosphere position for $M_\odot$ (§IX).

X.2. What remains open (stages 2 and 3 of the derivation)

The full derivation of the Einstein equation $G_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$ requires the following two stages, explicitly formulated in [15] §XIV.3 and not part of the task of the present paper:

1. **Stage 2 (source).** Derivation of the energy-momentum tensor $T_{\mu\nu}$ from the (B,I,S) structure of the observer through the SYNC projector $P_{O,\text{SYNC}}$ (with proof of idempotency — conjecture T_{idemp} [15] §XIV.2); a closed form $\chi_\Lambda(S^*)$ for the cosmological constant — conjecture $\text{T}_{\Lambda(S^*)}$ [15] §XIV.2. The connection with the thermodynamic derivation [8] provides an independent verification channel for this stage.
2. **Stage 3 (closure).** Proof of the field equation as the condition of Φ-self-consistency; the dynamical Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ as a Noether consequence of diffeomorphism invariance (conjecture $\text{T}_{\text{Bianchi}}$ [15] §XIV.2). The kinematic identity A.T3 of the present work is a necessary but not sufficient condition: the dynamical version requires a proof within the framework of a variational principle on the configuration manifold.

X.3. Links to the extended ODTOE corpus

The tensor apparatus of §III—§VII naturally combines with the extended ODTOE corpus:

- The connection ∇_μ (F3) uses Φ-iteration, the spectral properties of which and its fixed points are studied in [16] (the unified operator Φ). The stationarity of the Kerr metric in the region without external perturbations is equivalent to Φ-fixedness, which makes the tensor ansatz (F12) consistent with the equilibrium nature of Fix(Φ).

- The curvature $R^{\rho}{}_{\sigma\mu\nu}$ (F5) measures the SYNC defect along a closed contour; the dynamics of this defect over time are described by the equations on dB/dt from [17] §III, which provides a bridge to gravitational waves and non-stationary metrics.
- The Kerr ergosphere and horizon (8.2)—(8.3) give the limiting case of black-hole phenomenology [18]; the informational interpretation of the horizon as the boundary of accessibility to $\mathcal{C}$ for an external observer is preserved unchanged from [15] §IX.

XI. CONCLUSION

In the present work the tensor structure of gravity in ODTOE is built as a closed sequence: metric $g_{\mu\nu}$ as observer-correlator (F1) $\to$ covariant derivative ∇_μ as Φ-iteration commutator (F3) with Levi-Civita Christoffel symbols (F4) $\to$ Riemann tensor $R^{\rho}{}_{\sigma\mu\nu}$ as a measure of non-commutativity of SYNC operations (F5)—(F6) $\to$ Ricci tensor (F7), scalar R (F8), Einstein tensor $G_{\mu\nu}$ (F9) with the kinematic identity $\nabla_\mu G^{\mu\nu} = 0$ (F10). The Schwarzschild solution (F11) is recovered as an exact ODTOE vacuum solution; the Kerr solution in Boyer—Lindquist coordinates (F12) is derived as an ansatz with a vortex SYNC component (F13) whose horizons and ergosphere coincide with the standard theory without fitting. A 50-digit numerical demonstration reproduced the perihelion shift of Mercury (42.99 arcsec/century) and the position of the equatorial ergosphere $r_E^{\text{eq}} = 2M$ for the solar mass.

The main methodological result: the tensor geometry of GR is a *concrete configuration-space construction* in ODTOE, not an additional postulate. Metric, connection, curvature, and Einstein arise as properties of the self-observation map Φ on the configuration manifold $\mathcal{C}$; the standard tensor identities (5.1)—(5.3), (F10) are preserved as purely geometric consequences. This closes the first stage of the programme [15] §XIV.3 and leaves the derivation of $T_{\mu\nu}$ from the B-functional and the dynamical Bianchi identity as explicit next steps with their own structural conjectures $\mathrm{T}_{\text{idemp}}$, $\mathrm{T}_{\Lambda(S^*)}$, and $\mathrm{T}_{\text{Bianchi}}$ formulated in [15] §XIV.2.

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CONFLICT OF INTERESTS

The author declares no conflict of interests.

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Order remark. The bibliography is organized in three conceptual blocks: external classical sources of GR (1—12), then ODTOE corpus works (13—20). Within each block the order corresponds to the first mention in the text. The convention of a three-block order is explicitly fixed in [15] §L-35-ext.

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