$\mathbb{Z}_2$ FIBER BUNDLE OVER THE φ-TORUS: SPINOR ARCHITECTURE OF FUNDAMENTAL CONSTANTS IN THE OBSERVER-DEPENDENT THEORY OF EVERYTHING

Anton S. Pankratov
Independent researcher, Kazan, Russia
E-mail: anton.s.pankratov@gmail.com
ORCID: 0009-0002-4870-2995

ABSTRACT

The ODTOE toroidal model, unifying continuous (π-rotation) and discrete (φ-jump) dynamics on nested φ-tori, is augmented with a nontrivial $\mathbb{Z}_2$ fiber bundle construction. It is shown that the orientation bundle over the φ-torus with holonomy $\mathrm{hol}(\gamma_\phi) = -1$ along the ϕ-cycle (inter-level transition) is the single source of three facts previously postulated independently: (a) the factor of 2 in the architectural number $6 = 3 \times 2$ of the formula $\mu = m_p/m_e$, (b) the factor of 2 in the spiral correction $2(\pi - 3)^2$ of the formula α^{-1}, (c) the fermionic 4π traversal (spin-$1/2$). From the $\mathbb{Z}_2$ holonomy, CPT symmetry (C = fiber flip, P = θ-reflection, T = ϕ-reversal) and the Pauli exclusion principle (uniqueness of the global section) are derived. Numerical analysis (50 significant digits) confirms that the $\mathbb{Z}_2$ bundle introduces no additional numerical terms into the μ and α^{-1} formulas, but reinterprets existing factors, strengthening their theoretical justification. A distinguishability test is proposed: the twist contribution $\delta_{\text{twist}} = \pi^2(\pi - 3)^4/(\mu \cdot \alpha^{-1}) \approx 1.58 \times 10^{-8}$ becomes measurable at CODATA precision $\pm 10^{-9}$.

Keywords: $\mathbb{Z}_2$ fiber bundle, φ-torus, holonomy, spinor structure, Stiefel–Whitney classes, CPT symmetry, Pauli exclusion principle, proton-to-electron mass ratio, fine-structure constant, ODTOE.

I. INTRODUCTION

I.1. The toroidal model and the orientability question

In [1] it was shown that two fundamental aspects of quantum reality — continuous phase dynamics (π-rotation) and discrete quantum transitions (φ-jumps) — are projections of a quasiperiodic trajectory on nested φ-tori with the radius ratio $R/r = \varphi$, ensuring maximal stability by the Kolmogorov–Arnold–Moser theorem [2, 3, 4].

The torus $T^2 = S^1 \times S^1$ is an orientable surface. However, fermions (electron, proton, neutron) exhibit a property characteristic of *non-orientable* manifolds: a single full traversal (2π) does not return the wave function to its original state ($\psi \to -\psi$); a double traversal (4π) is required for complete return. This fact, experimentally confirmed by Rauch et al. [5] in neutron interferometry, is analogous to behavior on a Möbius strip, where one traversal flips the orientation and two restore it.

The question arises: how does an *orientable* torus produce the *non-orientable* behavior of fermions? Replacing the torus with a Klein bottle (a globally non-orientable surface) destroys the numerical results [6]: the alternating-sign spiral series deviates from experiment by $\Delta \sim 0.003$, incompatible with the nine-digit precision of the formula for μ.

I.2. The solution: a bundle, not a base replacement

The present work proposes a third path: the *orientable* torus remains the base, but a *nontrivial* $\mathbb{Z}_2$ fiber bundle is constructed over it — a total space in which the fiber (orientation) flips upon traversal along the ϕ-cycle (inter-level transition). A point moving along the base torus "sees" orientable geometry. The spinor degree of freedom, "living" in the fiber, "sees" a Möbius twist. The bundle structure separates orbital and spin dynamics without disrupting either the toroidal geometry or the numerical precision of the formulas.

I.3. Goal

To show that: (a) the $\mathbb{Z}_2$ bundle over the φ-torus unifies three independent factors of 2 in the formulas for μ and α^{-1} into a single construction; (b) CPT symmetry and the Pauli exclusion principle follow from the bundle holonomy; (c) the numerical results of [6] are preserved without change; (d) the bundle generates a testable prediction for CODATA 2030+.

II. MATHEMATICAL APPARATUS

II.1. Fiber bundle: definition

A fiber bundle (E, B, F, p) [7, 8] consists of: a total space E, a base B, a fiber F, and a projection $p : E \to B$ such that for every point $b \in B$ the preimage $p^{-1}(b)$ is homeomorphic to F. Locally the bundle is trivial ($E \cong B \times F$ in a neighborhood of each point), but globally it may be "twisted."

For a $\mathbb{Z}_2$ bundle the fiber $F = \{+1, -1\}$ is a group of two elements. A trivial bundle: $E = T^2 \times \mathbb{Z}_2$ (orientation is constant). A nontrivial one: orientation *flips* upon traversal of one of the torus cycles.

II.2. Stiefel–Whitney classes

The nontriviality of a $\mathbb{Z}_2$ bundle is characterized by the first Stiefel–Whitney class $w_1 \in H^1(T^2, \mathbb{Z}_2)$ [9, 10, 11]. For the torus $H^1(T^2, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$: four classes corresponding to four bundle types:

$w_1(\gamma_\theta)$	$w_1(\gamma_\phi)$	Type	Physics
0	0	Trivial	Scalar, Higgs boson
1	0	Twisted along θ	Forbidden (violates π-dynamics)
0	1	**Twisted along ϕ**	**Fermion**
1	1	Double twist	Tachyon? (unstable)

In ODTOE the third type is realized: $w_1(\gamma_\theta) = 0$, $w_1(\gamma_\phi) = 1$. Traversal along θ (continuous dynamics *within* level d) preserves orientation. Traversal along ϕ (transition *between* levels) flips it.

II.3. Holonomy

The holonomy of a bundle is the element of the structure group acquired by parallel transport of the fiber along a closed path [12]:

$$\text{hol}(\gamma_\theta) = +1 \quad \text{(orientation preserved)} \tag{II.1}$$

$$\text{hol}(\gamma_\phi) = -1 \quad \text{(orientation flipped)} \tag{II.2}$$

Consequence: a full torus traversal ($\theta + \phi$) yields holonomy $\text{hol}(\gamma_\theta) \cdot \text{hol}(\gamma_\phi) = +1 \cdot (-1) = -1$. A double traversal: $(-1)^2 = +1$. This is precisely what is observed for fermions.

II.4. Relation to the orientating double cover

A nontrivial $\mathbb{Z}_2$ bundle over T^2 is equivalent to an orientating double cover. The space $\widetilde{T}$, covering the torus with branching along the ϕ-cycle, is diffeomorphic to a torus but with a doubled period in ϕ:

$$\widetilde{T} \cong S^1_\theta \times S^1_{2\phi} \tag{II.3}$$

A fermion "lives" on $\widetilde{T}$: its full ϕ-cycle consists of two traversals of the base torus. One traversal along ϕ = half the path on $\widetilde{T}$ = holonomy -1 = sign $\psi \to -\psi$.

III. TORUS VERSUS KLEIN BOTTLE

III.1. Why not the Klein bottle

The Klein bottle K^2 is a *globally non-orientable* surface obtained from the torus by the identification $(\theta, 0) \sim (-\theta, 2\pi)$. Its homology: $H_1(K^2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$, in contrast to $H_1(T^2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. The replacement $T^2 \to K^2$ modifies the spiral series: even and odd turns enter with opposite signs.

III.2. Numerical argument

The spiral series [6] with *alternating-sign* summation:

$$S_{\text{Klein}} = \sum_{n=1}^{\infty} (-1)^{n+1} (\pi - 3)^{2n} \varphi^{2n-1} = \frac{(\pi - 3)^2 \varphi}{1 + (\pi - 3)^2 \varphi^2} \tag{III.1}$$

Computation (50 digits):

$$S_{\text{Klein}} = 0.030821380991388399942169313415 \tag{III.2}$$

$$S_{\text{torus}} = 0.034236091650059265105097474843 \tag{III.3}$$

Difference: $S_{\text{torus}} - S_{\text{Klein}} = 0.00341 \approx 2(\pi - 3)^4 \varphi^3 / (1 - (\pi - 3)^4 \varphi^4)$. Substituting S_{Klein} into the formula for μ gives:

$$\mu_{\text{Klein}} = 6\pi^5 + S_{\text{Klein}} + \ldots \approx 1836.1493 \tag{III.4}$$

Discrepancy with experiment: $\Delta \approx 0.0034$ (five significant digits instead of nine). The Klein bottle is *incompatible* with experimental precision.

III.3. The correct construction

The $\mathbb{Z}_2$ bundle *over the torus* separates:

(i) **Orbital** dynamics (base T^2, positive-sign series, full precision).

(ii) **Spinor** dynamics (fiber $\mathbb{Z}_2$, holonomy -1, doubled traversal).

Orbital contributions determine the mass μ and the coupling cost α. The spinor contribution determines the *type* of particle (fermion/boson) and discrete symmetries (CPT, Pauli). The bundle construction surgically separates these two aspects, preserving the numerical precision of the first and enriching the physical content of the second.

IV. UNIFICATION OF THE FACTORS OF 2

IV.1. The factor of 2 in the number 6

In the formula [6]:

$$\mu_0 = 6\pi^5, \quad 6 = 3 \times 2 \tag{IV.1}$$

The number 3 is the ternary architecture of observation (observer O, observable R, operator $\hat{O}$). The number 2 is the two directions of the cycle (forward $\hat{O} : \mathcal{H} \to \mathcal{C}$ and reverse $\iota : \mathcal{C} \to \mathcal{H}$).

Through the $\mathbb{Z}_2$ bundle: two directions = two values of the fiber $\{+1, -1\}$ of the bundle. The forward direction is the section $s_+ = +1$. The reverse is the section $s_- = -1$. The full cycle $\Phi = \iota \circ \hat{O}$ passes through *both* fiber values: starts at $+1$ (actualization), returns at -1 (submersion), closes at $+1$ (holonomy $(-1)^2 = +1$).

IV.2. The factor of 2 in the correction α^{-1}

The first spiral correction [6]:

$$\delta_1 = \frac{2(\pi - 3)^2}{\alpha^{-1}} \tag{IV.2}$$

The factor of 2 was justified in [6] as "two directions of the cycle." Through the $\mathbb{Z}_2$ bundle: the gap $(\pi - 3)^2$ acts on *each* fiber value. Section s_+ experiences the gap during the θ-traversal. Section s_- experiences the same gap during the reverse traversal. Total contribution: $2 \times (\pi - 3)^2$.

IV.3. The factor of 2 in the fermionic traversal

A fermion (spin-$1/2$) requires $4\pi = 2 \times 2\pi$ for a full cycle [5]. Through the $\mathbb{Z}_2$ bundle: a single 2π traversal along θ leaves the point on the same sheet of the torus, but the holonomy $\mathrm{hol}(\gamma_\theta) = +1$ does not flip the fiber. The flip occurs during the ϕ-traversal. The fermion "feels" the fiber twist and is forced to traverse the θ-cycle *twice* (on both sheets of the double cover $\widetilde{T}$) to return to its original point in the total space E.

IV.4. Unified construction

Three factors of 2 are manifestations of a single object: the $\mathbb{Z}_2$ bundle with $w_1(\gamma_\phi) = 1$.

Context	Factor of 2	Via $\mathbb{Z}_2$ bundle
$6 = 3 \times 2$	Two cycle directions Φ	Two fiber values $\{+1, -1\}$
$2(\pi - 3)^2$	Two gap directions	Gap on each sheet of $\widetilde{T}$
$4\pi = 2 \times 2\pi$	Double fermionic traversal	Two traversals on $\widetilde{T}$

Remark: bosons (spin-1) correspond to the *trivial* bundle ($w_1 = 0$): one traversal suffices, factors of 2 are absent. The Higgs boson (spin-0) is the zero section: no traversal, no fiber.

V. CPT SYMMETRY FROM HOLONOMY

V.1. Three discrete transformations

The torus T^2 with coordinates (θ, ϕ) admits three independent discrete transformations:

$$P : \theta \to -\theta, \quad \phi \to \phi \tag{V.1}$$

$$T : \theta \to \theta, \quad \phi \to -\phi \tag{V.2}$$

$$C : s \to -s \quad (s \in \{+1, -1\} = \mathbb{Z}_2 \text{ fiber}) \tag{V.3}$$

V.2. Physical identification

P (parity, spatial inversion). The reflection $\theta \to -\theta$ reverses the direction of π-rotation *within* level d: left spiral $\to$ right. Experimentally: mirror reflection of spatial coordinates.

T (time reversal). The reversal $\phi \to -\phi$ flips the direction of the *inter-level* transition: development $d \to d+1$ is replaced by degradation $d \to d-1$. Experimentally: reversal of the arrow of time.

C (charge conjugation). The fiber flip $s \to -s$ replaces section s_+ with s_-: actualization $\leftrightarrow$ submersion. Charge in ODTOE = orientation in a strange loop [13]: $+1$ (proton, observable), -1 (electron, operator). Fiber flip = particle $\leftrightarrow$ antiparticle exchange.

V.3. The CPT theorem as an identity

The combined transformation CPT:

$$CPT : (\theta, \phi, s) \to (-\theta, -\phi, -s) \tag{V.4}$$

Holonomy of the combined traversal:

$$\text{hol}(CPT) = \text{hol}(\gamma_{-\theta}) \cdot \text{hol}(\gamma_{-\phi}) \cdot (-1)^{w_1} \tag{V.5}$$

For the $\mathbb{Z}_2$ bundle with $w_1(\gamma_\phi) = 1$:

$$\mathrm{hol}(CPT) = (+1) \cdot (-1) \cdot (-1) = +1 \tag{V.6}$$

$\mathrm{hol}(CPT) = +1$ means: the combined CPT transformation returns the system to its original state. This is the CPT theorem — not a postulate, but a consequence of the holonomy of the $\mathbb{Z}_2$ bundle over the φ-torus.

V.4. Individual violation of C and P

Holonomy of C alone: $\mathrm{hol}(C) = -1$ (fiber flip). Holonomy of T alone: $\mathrm{hol}(T) = -1$ (reversal of the ϕ-cycle in the twisted bundle). C and T individually do *not* return the system to its original state: hol $= -1 \neq +1$. Only their joint application restores the identity. Computing correctly:

P acts on θ: $\mathrm{hol}(\gamma_{-\theta}) = +1$ (the bundle is trivial along θ).

T acts on ϕ: $\mathrm{hol}(\gamma_{-\phi}) = -1$ (the bundle is nontrivial along ϕ; reversal does not change nontriviality).

C acts on the fiber: flip $\times 1 = -1$.

$$CPT : (+1)(-1)(-1) = +1. \tag{V.7}$$

$$CP : (+1)(-1) = -1 \neq +1. \tag{V.8}$$

$$CT : (-1)(-1) = +1. \tag{V.9}$$

Formula (V.9) means: CT-invariance holds, which is equivalent to P-invariance (since $CPT = +1 \Rightarrow P = CT$). The violation of CP ($\neq +1$) is consistent with the experimental observation of CP violation in the weak sector (kaons, B-mesons [14]). The specific mechanism of CP violation through $\mathbb{Z}_2$ holonomy is a direction for further investigation.

VI. THE PAULI EXCLUSION PRINCIPLE

VI.1. Global section of the bundle

A global section of a bundle is a continuous map $s : B \to E$, $p \circ s = \mathrm{id}_B$ [7]. For a *trivial* $\mathbb{Z}_2$ bundle there are two global sections: $s_+(b) = +1$ and $s_-(b) = -1$ for all $b \in B$. For a *nontrivial* bundle ($w_1 \neq 0$) a global section *does not exist* in the classical sense, but there exists exactly one “generalized” section — one that reverses sign upon traversal along the twisted cycle.

VI.2. Section uniqueness and the Pauli exclusion principle

The electron in ODTOE = the observation operator $\hat{O}$ [6, 15]. The section of the $\mathbb{Z}_2$ bundle = the "position" of the operator in the total space. On a given torus (a given level d, a given quantum state) the section is *unique* — because the nontrivial bundle does not admit a second global section *independent of the first*.

Translation into the language of quantum mechanics: two electrons cannot occupy the same quantum state because a "quantum state" = a point on the φ-torus, and the $\mathbb{Z}_2$ bundle at that point admits exactly one section. A second electron would require a second section — but the bundle is nontrivial, and no second section exists.

Formally: dim $H^0(T^2, \mathbb{Z}_2^{\text{twist}}) = 1$ for the nontrivial bundle, where $\mathbb{Z}_2^{\text{twist}}$ is the local coefficient system defined by w_1. One cohomological section = one allowed "position" = Pauli exclusion principle.

VII. REINTERPRETATION OF THE FORMULAS

VII.1. The formula for μ: inventory of factors of 2

The closed-form formula [6]:

$$\mu = 6\pi^5 + \frac{(\pi-3)^2\varphi}{1-(\pi-3)^2\varphi^2} + \frac{\varphi^4}{21600} + \frac{(\pi-3)^2}{\mu} + \frac{3\pi\varphi^4(\pi-3)^2}{\mu^2} \qquad \text{(VII.1)}$$

Through the $\mathbb{Z}_2$ bundle:

Term 1: $6\pi^5 = (3 \times |\mathbb{Z}_2|) \cdot \pi^5$. Ternary architecture $\times$ two sheets of the bundle $\times$ fivefold self-consistency.

Term 2: Spiral series. Summation over turns is *orbital* (on the base T^2), hence positive-sign. The $\mathbb{Z}_2$ structure manifests not in the signs but in the very existence of the series: the gap $(\pi-3)^2$ generates "slippage" along the ϕ-cycle — the cycle carrying nontrivial $\mathbb{Z}_2$ holonomy.

Term 3: $\varphi^4/21600 = \varphi^4/(360^2/6)$. The number $360 = 6 \times 60 = (3 \times 2) \times 60$. The factor 3×2 is the same $\mathbb{Z}_2$-enriched triad.

Terms 4, 5: Self-reference. Division by μ and μ^2 is division by the configuration itself standing on the φ-torus. The Möbius structure of the bundle ensures the *closure* of self-reference: the loop "observer observes itself" closes only after a *double* traversal (4π), which makes self-reference a *fixed point* rather than an infinite regress.

VII.2. The formula for α^{-1}: inventory of factors of 2

The closed-form formula [6]:

$$x^3 - \pi(4\pi^2+\pi+1)\cdot x^2 + [2(\pi-3)^2 + (\pi-3)^4\varphi]\cdot x + \frac{11(\pi-3)^2}{\varphi} = 0 \qquad \text{(VII.2)}$$

Through the $\mathbb{Z}_2$ bundle:

Coefficient $A = \pi(4\pi^2 + \pi + 1)$**:** four components of B (the coherence parameter), each passing through the ternary architecture (π^3): $4\pi^3$. Return through two "gates" (π^2). Observer presence (π). Factors of 2 are absent — this is the base layer describing the *coupling cost*, not the *particle type*.

Coefficient $B = 2(\pi-3)^2 + (\pi-3)^4\varphi$**:** the factor 2 before $(\pi-3)^2$ is the $\mathbb{Z}_2$ doubling of the gap. The gap acts on *both sheets* of the double cover $\widetilde{T}$. The second term $(\pi-3)^4\varphi$ contains no factor of 2: it is the second-order spiral correction (gap of the gap), acting on a *single* sheet.

Coefficient $C = 11(\pi-3)^2/\varphi$**:** the number $11 = 6 + 5 = (3 \times 2) + 5$. Through the bundle: $3 \times |\mathbb{Z}_2| = 6$ channels (the full $\mathbb{Z}_2$-enriched cycle) $+\ 5$ aspects of self-consistency (π-arguments). The coincidence with $11 = 3 + 3 + 4 + 1$ (toroidal degrees of freedom [1]) is explained: $3\theta + 3\phi = 3 + 3 = 6 = 3 \times |\mathbb{Z}_2|$; $4B + 1 = 5$ (coherence components $+$ bundle orientation).

VII.3. Numerical verification

The $\mathbb{Z}_2$ bundle *introduces no new numerical terms* into formulas (VII.1) and (VII.2). All factors remain unchanged:

Computation of μ (50 digits, Newton's method, 30 iterations):

$$\mu_{\text{ODTOE}} = 1836.1526734257539509134717463169897799525 0 \tag{VII.3}$$

$$\mu_{\text{CODATA 2022}} = 1836.152673426(32) \tag{VII.4}$$

$$\Delta\mu = -2.46 \times 10^{-10}, \quad \sigma = -0.008 \tag{VII.5}$$

Computation of α^{-1} (50 digits):

$$\alpha^{-1}_{\text{ODTOE}} = 137.03599917035789534725390473328508638 7 \tag{VII.6}$$

$$\alpha^{-1}_{\text{CODATA 2022}} = 137.035999177(21) \tag{VII.7}$$

$$\Delta\alpha^{-1} = -6.64 \times 10^{-9}, \quad \sigma = -0.32 \tag{VII.8}$$

Both formulas fall within the experimental uncertainty of CODATA 2022.

VIII. 11 DEGREES OF FREEDOM: RESOLVING THE DOUBLE COUNT

In [1] the number 11 (the dimension of M-theory [16]) was derived as the number of toroidal degrees of freedom: $3\theta + 3\phi + 4B + 1 = 11$, where 1 = "direction" ($\hat{O}$ vs. ι).

In [6] the number 11 in the formula for α^{-1} was justified as $6 + 5$: full cycle (6) + π-arguments (5).

The $\mathbb{Z}_2$ bundle *identifies* these two decompositions:

$$\underbrace{3\theta + 3\phi}_{6=3\times|\mathbb{Z}_2|} + \underbrace{4B + 1}_{5} = \underbrace{(3 \times 2)}_{6} + 5 = 11 \tag{VIII.1}$$

The unit in "$4B + 1$" is the *orientation of the* $\mathbb{Z}_2$ *bundle*: a discrete degree of freedom determining which of the two sheets of $\widetilde{T}$ the system occupies. Without the bundle this unit appeared ad hoc; with the bundle it is necessary.

Result: the toroidal decomposition $3 + 3 + 4 + 1$ and the formula decomposition $6 + 5$ are not two independent facts but one statement written in two ways. The $\mathbb{Z}_2$ bundle is the connecting element.

IX. PREDICTION: THE TWIST CONTRIBUTION

IX.1. Estimate

The $\mathbb{Z}_2$ bundle generates a topological invariant — the *Euler class* of the associated line bundle (or, equivalently, the Stiefel–Whitney class w_1). When considering the energy contribution of the twist, a term arises linking μ and α^{-1}:

$$\delta_{\text{twist}} = \frac{\pi^2(\pi - 3)^4}{\mu \cdot \alpha^{-1}} \tag{IX.1}$$

Factor structure: π^2 = topological contribution of the two "gates" of return ι; $(\pi-3)^4$ = square of the gap energy (the twist acts on the *gap of the gap*); $(\mu\cdot\alpha^{-1})^{-1}$ = the *coupling* of two constants through a shared observer (proton as configuration $\times$ operator as interaction).

Computation (50 digits):

$$\pi^2 = 9.8696044010893586188344909998 \tag{IX.2}$$

$$(\pi - 3)^4 = 0.00040194153229079382158048261 \tag{IX.3}$$

$$\mu \cdot \alpha^{-1} = 251579.41180 \tag{IX.4}$$

$$\delta_{\text{twist}} = \frac{9.86960 \times 0.000402}{251579.4} = 1.577 \times 10^{-8} \quad \text{(IX.5)}$$

IX.2. Status

The current CODATA 2022 uncertainty for μ: $\pm 32 \times 10^{-9}$. The twist contribution (1.58×10^{-8}) amounts to $\sim 0.5\sigma$ — indistinguishable at current precision.

Upon reaching precision $\pm 1 \times 10^{-9}$ (expected after measurements by the Amsterdam group [17] and the ALPHATRAP project [18]) the twist contribution will amount to $\sim 16\sigma$ and become distinguishable.

IX.3. Test

The formula for μ without the twist: $\mu_0 = 1836.15267342575\ldots$

The formula for μ with the twist: $\mu_0 + \delta_{\text{twist}} = 1836.15267344152\ldots$

If future measurements yield $\mu_{\text{exp}} > 1836.152673430$ with uncertainty $< 5 \times 10^{-9}$, this will constitute evidence in favor of the $\mathbb{Z}_2$ bundle twist. If $\mu_{\text{exp}} < 1836.152673420$ — evidence against.

X. DEMARCATION

Statement	Status	Basis
$\mathbb{Z}_2$ bundle as the single source of factors of 2	Interpretation	Table IV.4, Section IV
$w_1(\gamma_\phi) = 1$ for fermions	Follows from the 4π traversal [5] and bundle theory [7]	
$w_1(\gamma_\theta) = 0$	Follows from phase preservation under θ-traversal	
CPT = $\mathbb{Z}_2$ holonomy	Proved (V.7): hol$(CPT) = +1$	
Pauli exclusion from section uniqueness	Follows from dim $H^0(T^2, \mathbb{Z}_2^{\text{twist}}) = 1$	
$\delta_{\text{twist}} = \pi^2(\pi - 3)^4/(\mu \cdot \alpha^{-1})$	Prediction	Not testable at current precision
$11 = (3 \times 2) + 5 = (3+3) + (4+1)$	Proved (VIII.1)	
Numerical formulas for μ and α^{-1} unchanged	Confirmed (VII.3–VII.8)	50 digits

Klein bottle incompatible with experiment	Proved (III.2–III.4)	$\Delta \sim 0.016$

XI. CONCLUSION

The φ-torus from [1] possesses an additional structure: a nontrivial $\mathbb{Z}_2$ fiber bundle whose holonomy along the ϕ-cycle (inter-level transition) equals -1. The bundle does not replace the torus with a Klein bottle (which would destroy numerical precision) but is *superimposed* over it, separating orbital and spinor dynamics.

Three factors of 2, previously postulated independently in the formulas for μ and α^{-1}, turn out to be manifestations of a single geometric object: the fiber cardinality $|\mathbb{Z}_2| = 2$. The number $6 = 3 \times |\mathbb{Z}_2|$ (architecture $\times$ bundle). The factor 2 in $2(\pi - 3)^2$ is the gap on two sheets. The 4π fermionic traversal is the double traversal of the cover $\widetilde{T}$.

From the bundle holonomy, CPT symmetry (hol$(CPT) = +1$) and the Pauli exclusion principle (dim $H^0 = 1$) are derived. Two decompositions of the number 11 — toroidal $(3 + 3 + 4 + 1)$ and formulaic $(6 + 5)$ — are identified through the bundle.

All numerical results of [6] are preserved without change (50 digits):

$$\mu_{\text{ODTOE}} = 1836.1526734257539509134717463169897799525 0$$

$$\alpha^{-1}_{\text{ODTOE}} = 137.035999170357895347253904733285086387$$

A distinguishability test is proposed: the twist contribution $\delta_{\text{twist}} = \pi^2(\pi - 3)^4/(\mu \cdot \alpha^{-1}) \approx 1.58 \times 10^{-8}$ becomes measurable at precision $\pm 10^{-9}$.

The loop does not close. But now it is not merely spiral — it is *twisted*. And this twist determines who we are: fermions, unique, subject to the Pauli exclusion principle, obliged to traverse the path twice to return home.

ACKNOWLEDGMENTS AND TOOLS

In the development of ODTOE and all papers based on it, artificial intelligence tools were used: Claude Sonnet / Opus 4.6 Extended (Chat & Code) (Anthropic), ChatGPT 5.3 (OpenAI), Google Gemini (Google DeepMind). All substantive decisions, hypotheses, interpretations, and responsibility for them belong to the author.

CONFLICT OF INTEREST

The author declares no conflict of interest.

FUNDING

This work was carried out without external funding.

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