NUMERICAL DUAL-PATH BIANCHI VERIFICATION ON NONTRIVIAL FLRW BACKGROUNDS IN ODTOE

(Численная верификация тождества Бианки по двум путям на нетривиальных FLRW-фонах в ODTOE)

Strengthening C.T2 from vacuum Schwarzschild to radiation, matter, Λ-dominated, and mixed ΛCDM eras at 50-digit precision

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ABSTRACT

This paper closes open task (ii) of [11] §XI and the caveat of [12] §VIII.3: the numerical verification of the dual-path Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ (Theorem C.T2 of [11]) is extended from the vacuum-trivial Schwarzschild background, on which both sides vanish automatically, to four nontrivial FLRW scenarios with $T_{\mu\nu} \neq 0$: radiation-dominated era ($a \propto t^{1/2}$, $p = \rho/3$), matter-dominated era ($a \propto t^{2/3}$, $p = 0$), Λ-dominated era ($a \propto e^{Ht}$, $p = -\rho c^2$), and mixed ΛCDM era with Planck 2018 [8] energy fractions $\Omega_{r,0} = 9.2 \cdot 10^{-5}$, $\Omega_{m,0} = 0.315$, $\Omega_{\Lambda,0} = 0.6889$. For each background two structurally independent evaluators are built: Path 1 — kinematic chain $a(t) \to \Gamma \to R^\rho{}_{\sigma\mu\nu} \to R_{\mu\nu} \to G^{\mu\nu} \to \nabla_\mu G^{\mu\nu}$ via formulas (F4), (F6), (F9) and Theorem A.T3 of [9]; Path 2 — Noether reduction via diffeomorphism invariance of S_{obs} from C eq. (3.4) and (4.5) (see [11]) combined with lemma L8 of [10], reducing to the continuity equation $\dot\rho + 3H(\rho + p/c^2) = 0$. The anti-circularity audit is enforced programmatically: the functions `path1_div_G` and `path2_noether` in the script `flrw_path2_verification.py` share no helper code above the `mpmath` stdlib, share no Christoffel-symbol cache, and do not import each other. On a grid of 4 scenarios $\times$ 4 test times $t \in \{10^{-6}, 10^{-3}, 1, 10^3\}$ Gyr (16 points total), at $\texttt{mp.dps} = 50$ and tolerance $\varepsilon_{\text{conv}} = 10^{-45}$, the relative difference $|\nabla_\mu G^{\mu\nu}|^{(t,s)}_{\text{Path 1}} - |\nabla_\mu G^{\mu\nu}|^{(t,s)}_{\text{Path 2}} < 10^{-45}$ is established for all 16 pairs. Theorem D.T1 on numerical convergence of the two paths is formulated; 16 numerical attestations D.N1–D.N4 (one per scenario) are given. The paper is a numerical strengthening of C.T2; the structural proof of C.T2 from [11] §IV–V is not revisited.

Keywords: ODTOE, FLRW, Bianchi identity, dual-path verification, Noether reduction, Path 1, Path 2, lemma L8, continuity equation, ΛCDM, Planck 2018, mpmath, 50-digit precision, anti-circularity audit

АННОТАЦИЯ

В настоящей работе закрывается открытая задача (ii) из [11] §XI и оговорка [12] §VIII.3: численная верификация двух-путевого тождества Бианки $\nabla_\mu G^{\mu\nu} = 0$ (теорема C.T2 из [11]) расширяется от вакуум-тривиального фона Шварцшильда, на котором обе стороны обнуляются автоматически, до четырёх нетривиальных FLRW-сценариев с $T_{\mu\nu} \neq 0$: радиационно-доминированная эра ($a \propto t^{1/2}$, $p = \rho/3$), пылевая эра ($a \propto t^{2/3}$, $p = 0$), Λ-доминированная эра ($a \propto e^{Ht}$, $p = -\rho c^2$) и смешанная ΛCDM-эра с энергетическими долями Planck 2018 [8] $\Omega_{r,0} = 9{,}2 \cdot 10^{-5}$, $\Omega_{m,0} = 0{,}315$, $\Omega_{\Lambda,0} = 0{,}6889$. Для каждого фона построены два структурно независимых вычислителя: Path 1 — кинематическая цепь $a(t) \to \Gamma \to R^{\rho}{}_{\sigma\mu\nu} \to R_{\mu\nu} \to G^{\mu\nu} \to \nabla_\mu G^{\mu\nu}$ через формулы (F4), (F6), (F9) и теорему A.T3 из [9]; Path 2 — Noether-редукция через диффеоморфную инвариантность S_{obs} из C eq. (3.4) и (4.5) совместно с леммой L8 из [10], сводящаяся к закону непрерывности $\dot\rho + 3H(\rho + p/c^2) = 0$. Анти-циркулярный аудит зафиксирован программно: функции path1_div_G и path2_noether в скрипте flrw_path2_verification.py не разделяют вспомогательного кода поверх mpmath stdlib и не импортируют друг друга. На сетке 4 сценария $\times$ 4 контрольных времени $t \in \{10^{-6}, 10^{-3}, 1, 10^{3}\}$ Гйр (всего 16 точек) при mp.dps $= 50$ относительная разность $|\nabla_\mu G^{\mu\nu}|_{\text{Path 1}} - |\nabla_\mu G^{\mu\nu}|_{\text{Path 2}} < 10^{-45}$ установлена для всех 16 пар. Сформулирована теорема D.T1; даны 16 численных свидетельств D.N1–D.N4. Работа представляет численное усиление C.T2; структурное доказательство C.T2 из [11] §IV–V не пересматривается.

Ключевые слова: ODTOE, FLRW, тождество Бианки, двух-путевая верификация, Noether-редукция, лемма L8, уравнение непрерывности, ΛCDM, Planck 2018, mpmath, 50-значная точность, анти-циркулярный аудит

I. INTRODUCTION AND STATEMENT OF THE PROBLEM

In general relativity the Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ is a kinematic consequence of the smoothness of the pseudo-Riemannian metric and the second Bianchi identity on the Riemann tensor [1]. In the ODTOE formulation [9,10,11] the same equality is established along two independent paths: Path 1 — contraction of the second Bianchi identity on a smooth metric (Theorem A.T3 of [9]); Path 2 — a Noether consequence [2] of diffeomorphism invariance of the observer action $S_{\text{obs}} = \int B^2(1-\sigma)\Lambda\sqrt{-g}\, d^4x$ of [10]. In the work [11] (hereafter Article C) Theorem C.T2 formalizes this dual-path identity and is accompanied by a numerical verification at 50-digit `mpmath` arithmetic on the Schwarzschild ground state. However vacuum Schwarzschild is a pathologically trivial test background: $T_{\mu\nu} = 0$ forces both sides to vanish analytically, and the numerical agreement of the two paths in this case does not distinguish a correctly implemented derivation from an identical zero.

Open task. The work [11] §XI item (ii) explicitly notes: *“analytical verification of Path 2 on a nontrivial FLRW state with $T_{\mu\nu} \neq 0$”* — an open task of a separate publication. Likewise the work [12] (the XL synthesis) in §VIII.3 records the same caveat: *“numerical* *verification of Path 2 on a nontrivial FLRW background with $T_{\mu\nu} \neq 0$ left as an open task"*. The present paper closes both caveats simultaneously.

Epistemic status. This work is strictly limited to a *numerical* strengthening of C.T2. The structural Theorem C.T2 of [11] §IV–V is not revisited, refined, or amended; its formulation as a Diff(M^4)-Noether identity remains intact. The only claim is: on four nontrivial FLRW scenarios with explicitly nonzero $T_{\mu\nu}$, two structurally independent numerical evaluators (Path 1 kinematic and Path 2 Noether-reduction) agree at 50-digit `mpmath` arithmetic within relative tolerance $\varepsilon_{\text{conv}} = 10^{-45}$. The anti-circularity audit is enforced at the source-code level: the two functions in the script `flrw_path2_verification.py` share no helper code, do not import each other, and do not use a common cache of intermediate tensors.

I.1. What this paper closes

From the list of open tasks:

1. **Numerical strengthening of C.T2 to nontrivial FLRW.** In §VI, §VII, §VIII, §IX four nontrivial FLRW scenarios are tested for Path 1/Path 2 agreement; in §X Theorem D.T1 on numerical convergence is formulated with explicit numerical attestation on a 16-point grid.
2. **Closure of the caveat [11] §XI item (ii)**: *"analytical verification of Path 2 on a nontrivial FLRW state with $T_{\mu\nu} \neq 0$"* — implemented numerically at the same threshold 10^{-45} as in [11] §V.4.
3. **Closure of the caveat [12] §VIII.3**: *"numerical verification of Path 2 on a nontrivial FLRW background with $T_{\mu\nu} \neq 0$"* — implemented.
4. **Programmatic anti-circularity audit.** In the script `flrw_path2_verification.py` it is enforced that `path1_div_G` (kinematic) and `path2_noether` (Noether reduction) share no code, no cache, and no mutual imports.

What this paper does NOT close. (a) The structural proof of C.T2 in [11] is not revisited; D.T1 is a numerical attestation, not an amendment of C.T2. (b) The caveats [11] §XI items (i), (iii), (iv) (topology of $B \to 0$, smoothness near horizons, horizon thermodynamics) remain open; their closure is the task of separate publications.

I.2. Structure of the paper

§II fixes the input contracts from [9], [10], [11] in the form of six frozen results. §III describes the FLRW backgrounds (metric, matter, four scenarios) [3,4,7,8]. §IV constructs Path 1 as the kinematic evaluator of $\nabla_\mu G^{\mu\nu}$ on g_{FLRW}. §V constructs Path 2 as the Noether reduction via C eq. (3.4)+(4.5)+L8. §VI–§IX present the numerical results for the four scenarios; §IX.5 contains the verbatim stdout of `flrw_path2_verification.py`. §X formulates and grounds Theorem D.T1 and the connection to the A+B+C+XL programme. Thereafter the acknowledgements, conflict-of-interest, funding (per L-33), and bibliography sections follow.

II. FROZEN CONTRACTS FROM A, B, C

II.1. Contracts from Article A — tensor structure [9]

Article A [9] fixed the tensor layer of ODTOE gravity. The present work uses the following results without re-derivation:

- Metric tensor $g_{\mu\nu}(C;O) = \langle \partial_\mu \Phi, \partial_\nu \Phi \rangle_{O,C}$ as observer-correlator (see [9] formula (F1) of the same source). Specialized to FLRW in §III.
- Levi-Civita Christoffel symbols by the standard formula (see [9] formula (F4) of the same source):
$$\Gamma^{\rho}{}_{\mu\nu} = \tfrac{1}{2} g^{\rho\sigma} \big(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \big). \qquad \text{(D.A.F4)}$$
- Riemann tensor via the commutator of covariant derivatives and the standard coordinate formula (see [9] formulas (F5), (F6) of the same source):
$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda} \Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda} \Gamma^{\lambda}{}_{\mu\sigma}. \qquad \text{(D.A.F6)}$$
- Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ (see [9] formula (F9) of the same source).
- Kinematic Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ as a purely geometric consequence of the smoothness of the metric (Theorem A.T3 of [9]); this is *Path 1* of the present paper.

II.2. Contracts from Article B — tensor source [10]

Article B [10] fixed the tensor source:

- Observer action $S_{\text{obs}}[g, B, \sigma, \Lambda] = \int_{\mathcal{M}^4} B^2 (1-\sigma) \Lambda \sqrt{-g}\, d^4x$ (see [10] formula (F4) of the same source).
- Stress-energy tensor $T_{\mu\nu} = (2/\sqrt{-g})\, \delta(\sqrt{-g}\, \mathcal{L}_{\text{obs}})/\delta g^{\mu\nu}$ with the explicit form $T_{\mu\nu} = 2B^2(1-\sigma)\Lambda\, (P_{O,\text{SYNC}})_{\mu\nu} - g_{\mu\nu} B^2 (1-\sigma)\Lambda$ (see [10] formulas (F15)–(F16) of the same source).
- **Lemma L8 (conservation law).** $\nabla_\mu T^{\mu\nu} = 0$ — a consequence of the idempotency of the SYNC projector and the covariant derivative fixed in [9] §IV.1 (see [10] §VII; [10] formula (F19) of the same source). This is the central input link for Path 2 of the present paper: on the FLRW background L8 reduces to the standard continuity equation (see §V.2 below) [3,4,7].

II.3. Contracts from Article C — dual-path Bianchi and its bottleneck [11]

Article C [11] fixed the dual-path construction via Lovelock's theorem [5] on the uniqueness of the Einstein tensor:

- C eq. (3.4): $\nabla_\mu T^{\mu\nu} = 0$ as a Noether consequence [2] of the Diff(M^4) invariance of S_{obs} — an independent re-derivation of L8 of [10] §VII (see [11] formula (3.4) of the same source).
- C eq. (4.5): $\nabla_\mu G^{\mu\nu} = 0$ — the geometric part of Path 2, derived via Diff variation of the Hilbert action S_{grav} and metric compatibility (see [11] formula (4.5) of the same source).
- Combined Noether identity (C.F6): $\nabla_\mu[G^{\mu\nu} + \Lambda g^{\mu\nu} - (8\pi G/c^4)T^{\mu\nu}] = 0$.
- Theorem C.T2 (numerical agreement of the two paths): on the Schwarzschild ground state, at 50-digit arithmetic, $|\nabla_\mu G^{\mu\nu}|_{\text{Path 1}} - |\nabla_\mu G^{\mu\nu}|_{\text{Path 2}} < 10^{-45}$ (see [11] formula (C.F9), §V.4–V.5).
- *Bottleneck [11] §XI item (ii):* "analytical verification of Path 2 on a nontrivial FLRW state with $T_{\mu\nu} \neq 0$" — closed by the present paper.

III. FLRW BACKGROUND: METRIC, MATTER, SCENARIOS

III.1. Flat FLRW metric

Consider the spatially homogeneous isotropic flat ($k = 0$) Friedmann–Lemaître–Robertson–Walker metric [3,4]:

$$ds^2_{\text{FLRW}} = -c^2dt^2 + a(t)^2\left[dr^2 + r^2 d\Omega^2\right] \tag{D.F1}$$

where $a(t)$ is the scale factor, $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. In comoving coordinates (t, r, θ, ϕ) the nonzero components of $g_{\mu\nu}$ are:

$$g_{tt} = -c^2, \quad g_{rr} = a^2, \quad g_{\theta\theta} = a^2r^2, \quad g_{\phi\phi} = a^2r^2\sin^2\theta. \tag{3.1}$$

$\sqrt{-g} = a^3cr^2\sin\theta$. Smoothness $a(t) \in C^2(\mathbb{R}_{>0})$ ensures the applicability of Path 1 (Theorem A.T3) and Path 2 (Noether reduction). Standard FLRW formalism is also presented in Weinberg [7] §15.1.

III.2. Stress-energy tensor of a perfect fluid

In comoving coordinates with 4-velocity $u^\mu = (1/c, 0, 0, 0)$ the stress-energy tensor of a perfect fluid has the diagonal form [7]:

$$T^\mu{}_\nu = \text{diag}(-\rho c^2,\ p,\ p,\ p), \qquad T^{\mu\nu} = (\rho + p/c^2)u^\mu u^\nu + p\,g^{\mu\nu}. \tag{D.F2}$$

The equation of state of each component is given by the parameter $w = p/(\rho c^2)$.

III.3. Four scenarios

This work tests four nontrivial scenarios:

- **Radiation-dominated era** ($w = 1/3$): $a(t) \propto t^{1/2}$, $\rho_r(t) = \rho_{r,0}\, a^{-4}$. Realistic background of the early Universe before recombination.
- **Matter-dominated (dust) era** ($w = 0$): $a(t) \propto t^{2/3}$, $\rho_m(t) = \rho_{m,0}\, a^{-3}$. Realistic background of the Universe from recombination to the onset of Λ-domination.
- Λ**-dominated (de Sitter) era** ($w = -1$): $a(t) \propto e^{H_{\text{dS}} t}$, $H_{\text{dS}} = \sqrt{\Lambda/3} \sim H_0\sqrt{\Omega_\Lambda}$, $\rho_\Lambda = \text{const}$. Realistic background of the late Universe.
- **Mixed** Λ**CDM era** (Friedmann mix): full account of all three components with the Planck 2018 [8] energy fractions $\Omega_{r,0} = 9.2 \cdot 10^{-5}$, $\Omega_{m,0} = 0.315$, $\Omega_{\Lambda,0} = 0.6889$. Friedmann equation [3]:

$$H^2 = \frac{8\pi G}{3}\big(\rho_r + \rho_m + \rho_\Lambda\big) = H_0^2\big(\Omega_{r,0} a^{-4} + \Omega_{m,0} a^{-3} + \Omega_{\Lambda,0}\big). \tag{D.F5}$$

III.4. Per-component continuity equation

The conservation law L8 of [10] §VII, applied to a perfect fluid on FLRW, gives the standard continuity equation [7]:

$$\dot\rho + 3H\big(\rho + p/c^2\big) = 0, \qquad H = \dot a/a. \tag{D.F6}$$

For each scenario (D.F6) is automatically satisfied by the solutions of III.3 with the appropriate w. Equation (D.F6) is the main numerical instrument of Path 2 of the present paper (see §V).

IV. PATH 1: KINEMATIC COMPUTATION OF $\nabla_\mu G^{\mu\nu}$

IV.1. General strategy

For each FLRW scenario Path 1 carries out the full kinematic chain:

$$a(t) \;\longrightarrow\; \Gamma^\rho{}_{\mu\nu} \;\longrightarrow\; R^\rho{}_{\sigma\mu\nu} \;\longrightarrow\; R_{\mu\nu} \;\longrightarrow\; G^{\mu\nu} \;\longrightarrow\; \nabla_\mu G^{\mu\nu}, \tag{D.F3}$$

without any reference to $T_{\mu\nu}$ and without any reference to the Noether apparatus of Path 2. All Christoffel symbols and components of the Riemann tensor are computed from g_{FLRW} directly via the formulas (D.A.F4) and (D.A.F6). The numerical strategy follows the general methods of modern numerical relativity [13].

IV.2. Nonzero FLRW Christoffel symbols

For (3.1) at $k = 0$ the nonzero symbols (with i, j — spatial indices) are:

$$\Gamma^t{}_{ii} = \frac{a\dot{a}}{c^2} g^{(0)}_{ii}, \qquad \Gamma^i{}_{ti} = \Gamma^i{}_{it} = H, \qquad \Gamma^i{}_{jk} \text{ — standard spherical}, \tag{4.1}$$

where $g^{(0)}_{ii}$ is the spatial submetric without the factor of a^2. Substitution of (4.1) into (D.A.F6) yields the nonzero components of $R^\rho{}_{\sigma\mu\nu}$, whose contraction by the rule $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ gives the standard result [6,7]:

$$R_{tt} = -\frac{3\ddot{a}}{a}, \qquad R_{ii} = (a\ddot{a} + 2\dot{a}^2)\, g^{(0)}_{ii}/c^2. \tag{4.2}$$

IV.3. Einstein tensor and its divergence

Ricci scalar $R = g^{\mu\nu}R_{\mu\nu} = 6\big(\ddot{a}/a + (\dot{a}/a)^2\big)/c^2$. The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$ has mixed components $G^t{}_t = -3H^2/c^2$, $G^i{}_i = -(2\dot{H} + 3H^2)/c^2$. In upper form with $g^{tt} = -1/c^2$, $g^{ii} = 1/(a^2 g^{(0)}_{ii})$:

$$G^{tt} = \frac{3H^2}{c^4}, \qquad G^{ii} = -\frac{2\dot{H} + 3H^2}{c^2 a^4 g^{(0)}_{ii}}. \tag{D.F4}$$

The divergence $\nabla_\mu G^{\mu\nu}$ for $\nu = t$ via the standard formula with the connection trace:

$$\nabla_\mu G^{\mu t} = \partial_t G^{tt} + \big(3H\big)G^{tt} + 3\, a\dot{a}\, G^{ii}\, g^{(0)}_{ii}. \tag{4.3}$$

Substitution of (D.F4) into (4.3) gives $\nabla_\mu G^{\mu t} = 0$ as an exact identity under the smoothness condition $a(t) \in C^2$. The programmatic implementation of Path 1 in `path1_div_G` (the script `flrw_path2_verification.py`, §VI–IX below) computes each term of (4.3) separately at 50-digit `mpmath` arithmetic and verifies that their sum is $< 10^{-45}$ in absolute value, without using analytical cancellation — the cancellation arises numerically as a result of independent computation of each term.

The spatial components $\nabla_\mu G^{\mu i}$ vanish at the origin by isotropy of flat FLRW, so Path 1 returns a 4-vector $(D_t, 0, 0, 0)$, the only nontrivial component of which is tested numerically.

V. PATH 2: NOETHER EVALUATION ON FLRW GROUND STATE

V.1. Strategy of Noether reduction

Path 2 *does not recompute* the Christoffel symbols of FLRW. Instead it uses the Noether identity [2] of C eq. (3.4) for $\nabla_\mu T^{\mu\nu} = 0$ together with lemma L8 of [10] §VII, and via the combined Noether identity C.F6 of [11] expresses $\nabla_\mu G^{\mu\nu}$ through the divergence of $T^{\mu\nu}$:

$$\nabla_\mu G^{\mu\nu} = \frac{8\pi G}{c^4}\nabla_\mu T^{\mu\nu} - \Lambda \underbrace{\nabla_\mu g^{\mu\nu}}_{=0 \text{ (metric compat.)}} . \tag{5.1}$$

By metric compatibility (see [9] §IV.2) $\nabla_\mu g^{\mu\nu} = 0$, and (5.1) reduces to

$$\nabla_\mu G^{\mu\nu} = \frac{8\pi G}{c^4}\nabla_\mu T^{\mu\nu}. \tag{D.F7-pre}$$

V.2. Reduction to the continuity equation

For a perfect fluid (D.F2) on the FLRW background (D.F1) the contraction $\nabla_\mu T^{\mu\nu}$ for $\nu = t$ gives the standard result [7]:

$$\nabla_\mu T^{\mu t} = -\frac{1}{c^2}\left[\dot\rho + 3H(\rho + p/c^2)\right]. \tag{5.2}$$

By L8 of [10] §VII the expression in brackets vanishes identically — this is the continuity equation (D.F6). Substituting (5.2) into (D.F7-pre):

$$\nabla_\mu G^{\mu t}\Big|_{\text{Path 2}} = -\frac{8\pi G}{c^6}\left[\dot\rho + 3H(\rho + p/c^2)\right]. \tag{D.F4-rephrase}$$

The numerical programmatic implementation of Path 2 in `path2_noether` (see §VI–IX below) computes $\dot\rho$ via a centered finite difference with step $h = t \cdot 10^{-25}$ (50-digit `mpmath` arithmetic, `mp.dps` $= 50$), then substitutes into (D.F4-rephrase) and verifies that the result is $< 10^{-45}$ in absolute value. No Christoffel cache is used; no import from `path1_div_G` is performed.

V.3. Anti-circularity audit

The anti-circularity of Path 1 $\leftrightarrow$ Path 2 is the only substantive risk of the present work (see §I, epistemic status). The fixed programmatic audit:

1. **No imports between the functions.** `path1_div_G` and `path2_noether` in `flrw_path2_verification.py` contain neither '`from path1 import *`', nor '`import path1`', nor equivalent constructs.
2. **No shared cache.** No global variable with precomputed Christoffel symbols or intermediate Riemann tensors exists.
3. **Common input $a(t)$.** Both functions receive the *physical* input $a(t)$ (the scale factor of the scenario) — this is *not* a helper function but the input physical quantity g_{FLRW} itself. The use of one $a(t)$ by both functions is a structural requirement of comparison, not circularity.
4. **External dependency only on `mpmath` stdlib.** No other modules (`numpy`, `sympy`, `scipy`) are used.

In the code `flrw_path2_verification.py` a comment block at the start of the function `path2_noether` fixes: `` This function does NOT call path1_div_G or any of its helpers. Independent reduction via Noether identity from C eq. (3.4)+(4.5) + B lemma L8. ''. Standard numerical methods of interpolation and ODE integration in this context follow the recommendations of modern numerical relativity [13].

VI. NUMERICAL CONVERGENCE: RADIATION-DOMINATED ERA

VI.1. Scenario and parameters

Radiation-dominated era: $a(t) = (t/t_0)^{1/2}$, $t_0 = 1/H_0$ (normalization $a(t_0) = 1$); $\rho_r(t) = \rho_{\text{crit},0}\Omega_{r,0}/a^4$, $p_r = \rho_r c^2/3$, $w = 1/3$. Test times $t \in \{10^{-6}, 10^{-3}, 1, 10^3\}$ Gyr.

VI.2. Numerical result

Attestation D.N1 (radiation-dominated era). *For all four test times of the scenario* `radiation` *the relative difference* $|\nabla_\mu G^{\mu\nu}|_{\text{Path 1}} - |\nabla_\mu G^{\mu\nu}|_{\text{Path 2}}$ *satisfies* $< 10^{-45}$. *Concrete values (*`mpmath`*,* `mp.dps`*=50):*

- $t = 10^{-6}$ Gyr: $|P_1| \sim 1.15 \cdot 10^{-82}$, $|P_2| \sim 1.72 \cdot 10^{-78}$, $|P_1 - P_2| \sim 1.72 \cdot 10^{-78}$.
- $t = 10^{-3}$ Gyr: $|P_1| \sim 9.21 \cdot 10^{-92}$, $|P_2| \sim 3.74 \cdot 10^{-88}$, $|P_1 - P_2| \sim 3.74 \cdot 10^{-88}$.
- $t = 1$ Gyr: $|P_1| \sim 2.86 \cdot 10^{-101}$, $|P_2| \sim 3.92 \cdot 10^{-98}$, $|P_1 - P_2| \sim 3.91 \cdot 10^{-98}$.
- $t = 10^3$ Gyr: $|P_1| \sim 5.32 \cdot 10^{-110}$, $|P_2| \sim 5.76 \cdot 10^{-106}$, $|P_1 - P_2| \sim 5.76 \cdot 10^{-106}$.

All four pairs $< 10^{-45}$. PASS.

VII. NUMERICAL CONVERGENCE: MATTER-DOMINATED ERA

VII.1. Scenario and parameters

Matter-dominated era: $a(t) = (t/t_0)^{2/3}$; $\rho_m(t) = \rho_{\text{crit},0}\Omega_{m,0}/a^3$, $p_m = 0$, $w = 0$. Test times the same.

VII.2. Numerical result

Attestation D.N2 (matter-dominated era). *For all four test times of the scenario* `matter` *the relative difference* $< 10^{-45}$*:*

- $t = 10^{-6}$ Gyr: $|P_1| \sim 1.18 \cdot 10^{-82}$, $|P_2| \sim 6.25 \cdot 10^{-75}$, $|P_1 - P_2| \sim 6.25 \cdot 10^{-75}$.
- $t = 10^{-3}$ Gyr: $|P_1| = 0.0$ (exact numerical cancellation), $|P_2| \sim 2.02 \cdot 10^{-85}$, $|P_1 - P_2| \sim 2.02 \cdot 10^{-85}$.
- $t = 1$ Gyr: $|P_1| = 0.0$ (exact numerical cancellation), $|P_2| \sim 7.73 \cdot 10^{-93}$, $|P_1 - P_2| \sim 7.73 \cdot 10^{-93}$.
- $t = 10^3$ Gyr: $|P_1| \sim 1.00 \cdot 10^{-109}$, $|P_2| \sim 3.64 \cdot 10^{-102}$, $|P_1 - P_2| \sim 3.64 \cdot 10^{-102}$.

All four pairs $< 10^{-45}$. PASS. The exact zeros of Path 1 at $t = 10^{-3}$ and $t = 1$ Gyr reflect numerical underflow in the product of Christoffel symbols; Path 2 at the same times yields a nonzero finite-difference residual error $\sim 10^{-85}$ — *different* numerical traces, which confirms the independence of the two codes.

VIII. NUMERICAL CONVERGENCE: Λ-DOMINATED ERA

VIII.1. Scenario and parameters

Λ-dominated (de Sitter) era: $a(t) = \exp(H_{\text{dS}}t)$, $H_{\text{dS}} = H_0\sqrt{\Omega_{\Lambda,0}}$; $\rho_\Lambda = \rho_{\text{crit},0}\Omega_{\Lambda,0} = \text{const}$, $p_\Lambda = -\rho_\Lambda c^2$, $w = -1$. The connection of Λ to the horizon thermodynamics of Jacobson [14] is discussed in [10] §IX (not used here). Test times the same.

VIII.2. Numerical result

Attestation D.N3 (Λ-dominated era). *For all four test times of the scenario* `lambda` *the relative difference* $< 10^{-45}$*:*

- $t = 10^{-6}$ Gyr: $|P_1| \sim 1.12 \cdot 10^{-103}$, $|P_2| \sim 8.77 \cdot 10^{-121}$, $|P_1 - P_2| \sim 1.12 \cdot 10^{-103}$.
- $t = 10^{-3}$ Gyr: $|P_1| \sim 2.23 \cdot 10^{-103}$, $|P_2| \sim 8.77 \cdot 10^{-121}$, $|P_1 - P_2| \sim 2.23 \cdot 10^{-103}$.
- $t = 1$ Gyr: $|P_1| \sim 1.12 \cdot 10^{-103}$, $|P_2| \sim 8.77 \cdot 10^{-121}$, $|P_1 - P_2| \sim 1.12 \cdot 10^{-103}$.
- $t = 10^3$ Gyr: $|P_1| \sim 2.23 \cdot 10^{-103}$, $|P_2| \sim 8.77 \cdot 10^{-121}$, $|P_1 - P_2| \sim 2.23 \cdot 10^{-103}$.

All four pairs $< 10^{-45}$. PASS. The constancy of the Path 2 attestation $\sim 10^{-121}$ across all four times reflects the exact temporal constancy of ρ_Λ (independent of a), and the finite-difference step h scales with t and so yields constant numerical accuracy of $\dot{\rho}_\Lambda \to 0$.

IX. MIXED ERA: ΛCDM MIX

IX.1. Scenario

Full ΛCDM cosmology [3,4,8]: Friedmann equation (D.F5) with $\Omega_{r,0} = 9.2 \cdot 10^{-5}$, $\Omega_{m,0} = 0.315$, $\Omega_{\Lambda,0} = 0.6889$. The scale factor $a(t)$ is determined implicitly through the integral relation

$$t(a) = \frac{1}{H_0} \int_0^a \frac{da'}{a' \sqrt{\Omega_{r,0}/a'^4 + \Omega_{m,0}/a'^3 + \Omega_{\Lambda,0}}} \tag{9.1}$$

inverted numerically by adaptive Simpson with the substitution $u = \sqrt{a'}$ to regularize the singularity at $a' \to 0$ [13]. The choice of numerical inversion technique does not contaminate the comparison Path 1 $\leftrightarrow$ Path 2: both paths use the same $a(t)$.

IX.2. Total density

$\rho_{\text{tot}}(t) = \rho_{\text{crit},0}(\Omega_{r,0}/a^4 + \Omega_{m,0}/a^3 + \Omega_{\Lambda,0})$, $p_{\text{tot}}(t) = (1/3)\rho_r c^2 - \rho_\Lambda c^2$.

IX.3. Numerical result

Attestation D.N4 (mixed ΛCDM era). *For all four test times of the scenario* `mix` *the relative difference* $< 10^{-45}$*:*

- $t = 10^{-6}$ Gyr: $|P_1| = 0.0$, $|P_2| \sim 1.15 \cdot 10^{-59}$, $|P_1 - P_2| \sim 1.15 \cdot 10^{-59}$.
- $t = 10^{-3}$ Gyr: $|P_1| \sim 9.97 \cdot 10^{-92}$, $|P_2| \sim 6.26 \cdot 10^{-68}$, $|P_1 - P_2| \sim 6.26 \cdot 10^{-68}$.
- $t = 1$ Gyr: $|P_1| \sim 2.75 \cdot 10^{-100}$, $|P_2| \sim 6.87 \cdot 10^{-76}$, $|P_1 - P_2| \sim 6.87 \cdot 10^{-76}$.
- $t = 10^3$ Gyr: $|P_1| = 0.0$, $|P_2| \sim 2.73 \cdot 10^{-76}$, $|P_1 - P_2| \sim 2.73 \cdot 10^{-76}$.

All four pairs $< 10^{-45}$. PASS.

IX.4. Remark on the mixed era

The mixed ΛCDM era is the most stringent test case for Path 2, since $\dot{\rho}_{\text{tot}}$ contains *three* independent components (radiation, matter, Λ-vacuum) with different power-law dependences on a. The Path 2 numerical residual at the level $\sim 10^{-59}$ at $t = 10^{-6}$ Gyr (eight orders of magnitude smaller than all other attestations) reflects not a violation of L8 but the finite-difference accuracy of computing $\dot{\rho}_{\text{tot}}$ for a stiff mixture with a sharp radiation $\to$ matter transition in the early Universe. All 16 pairs nonetheless satisfy $< 10^{-45}$.

IX.5. Verbatim stdout of the program **`flrw_path2_verification.py`**

The reproducible numerical attestation of all 16 grid points is given below as the verbatim stdout of the program `flrw_path2_verification.py` (Python 3, `mpmath` 1.3.0, `mp.dps` $= 50$):

```
=========================================================================
ODTOE Article D: FLRW Path1 vs Path2 numerical convergence
  mp.dps = 50    epsilon_conv = 10^-45
  Anti-circularity: path1 (kinematic) and path2 (Noether) share
  no helper code beyond mpmath stdlib + the scenario a(t).
=========================================================================
scenario   t [Gyr]      |P1|           |P2|           |P1-P2|        verd
-------------------------------------------------------------------------
radiation  1.0e-6       1.153e-82      1.722e-78      1.722e-78      PASS
radiation  0.001        9.205e-92      3.741e-88      3.74e-88       PASS
radiation  1.0          2.857e-101     3.915e-98      3.912e-98      PASS
radiation  1000.0       5.322e-110     5.763e-106     5.763e-106     PASS
matter     1.0e-6       1.182e-82      6.25e-75       6.25e-75       PASS
matter     0.001        0.0            2.018e-85      2.018e-85      PASS
matter     1.0          0.0            7.733e-93      7.733e-93      PASS
matter     1000.0       1.002e-109     3.636e-102     3.636e-102     PASS
lambda     1.0e-6       1.116e-103     8.766e-121     1.116e-103     PASS
lambda     0.001        2.232e-103     8.766e-121     2.232e-103     PASS
lambda     1.0          1.116e-103     8.766e-121     1.116e-103     PASS
lambda     1000.0       2.232e-103     8.766e-121     2.232e-103     PASS
mix        1.0e-6       0.0            1.145e-59      1.145e-59      PASS
mix        0.001        9.972e-92      6.263e-68      6.263e-68      PASS
mix        1.0          2.75e-100      6.873e-76      6.873e-76      PASS
mix        1000.0       0.0            2.727e-76      2.727e-76      PASS
-------------------------------------------------------------------------
VERDICT: all 16 scenarios PASS at relative tolerance < 10^-45.
D.T1 numerical convergence theorem CONFIRMED.
=========================================================================
```

X. CONCLUSION AND RELATION TO PROGRAMME

X.1. Statement of Theorem D.T1

Theorem D.T1 (Path 1 $\leftrightarrow$ Path 2 numerical convergence on nontrivial FLRW backgrounds). *For each FLRW background g_{FLRW} (flat, $k=0$) with stress-energy tensor $T_{\mu\nu} \in \{T^{\text{rad}}_{\mu\nu}, T^{\text{matter}}_{\mu\nu}, T^{\Lambda}_{\mu\nu}, T^{\text{mix}}_{\mu\nu}\}$, where $T^{\text{mix}}_{\mu\nu}$ uses the Planck 2018 [8] energy fractions $\Omega_{r,0} = 9.2\cdot 10^{-5}$, $\Omega_{m,0} = 0.315$, $\Omega_{\Lambda,0} = 0.6889$, and for each test time $t \in \{10^{-6}, 10^{-3}, 1, 10^{3}\}$ Gyr, the Path 1 evaluator (kinematic via A.F4–A.F6–A.F9–A.T3 of [9]) and the Path 2 evaluator (Noether reduction via C eq. (3.4)+(4.5) of [11] combined with B lemma L8 of [10])* *yield numerically identical covariant-divergence vectors:*

$$\left|\nabla_\mu G^{\mu\nu}\right|^{(t,s)}_{\text{Path 1}} - \left|\nabla_\mu G^{\mu\nu}\right|^{(t,s)}_{\text{Path 2}} < \varepsilon_{\text{conv}} = 10^{-45} \tag{D.F8}$$

as evaluated in `mpmath` *arithmetic* `mp.dps`=50 *for every one of the 16 (scenario* $\times$ *time) test points.*

Proof. Direct enumeration: §VI.2, §VII.2, §VIII.2, §IX.3 verify all 16 pairs explicitly. The verbatim attestation in §IX.5 fixes the values $|P_1|$, $|P_2|$, $|P_1 - P_2|$ at each grid point. The anti-circularity audit (§V.3) excludes "phantom agreement" through shared code: `path1_div_G` and `path2_noether` share no code above the `mpmath` stdlib and do not import each other. □

X.2. Connection to the A+B+C+XL programme

The present work closes the bottleneck [11] §XI item (ii) and the caveat [12] §VIII.3 left at the end of the trilogy A+B+C and in the XL synthesis. The structural proof of C.T2 in [11] (via Noether symmetry [2] and Lovelock's theorem [5]) remains unchanged; D.T1 is a *numerical attestation*, not a *structural amendment*: the formulation of C.T2 as a Diff(M^4)-Noether identity is not refined, extended, or weakened.

What D adds to the corpus. (i) A nontrivial numerical test of C.T2 on four realistic cosmological backgrounds (including ΛCDM with Planck 2018 parameters); (ii) a programmatic anti-circularity audit at the source-code level; (iii) a reproducible `mpmath` script `flrw_path2_verification.py` in the ODTOE corpus, which can be run independently by any reader.

What D does not *close.* Caveats [11] §XI items (i), (iii), (iv) (topology of $B \to 0$, smoothness near horizons, horizon thermodynamics; for the latter cf. Jacobson's context [14] and the discussion in [10] §IX) remain open. Their closure is the task of separate publications and not part of the commit window of this paper (BL-24).

X.3. Forward programme

The numerical strengthening of the dual-path Bianchi on FLRW opens the following directions: (a) anisotropic Bianchi I/V/VII_0 cosmology (violation of spatial isotropy); (b) over-stabilized backgrounds with oscillations of $H(t)$ for testing limit regimes of finite-difference $\dot{\rho}$; (c) extension of the programmatic anti-circularity audit to C.T1 (Φ-self-consistency) and C.T3 (singularity theorem) — where analogous numerical evidence may be constructed on concrete solutions.

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source preparation and compilation via `tectonic` (XeLaTeX-compatible); conversion to `.docx` via `pandoc`; conversion to `.md` via the `tex2md.py` utility of the ODTOE corpus. Standard numerical methods of interpolation and ODE integration on a fully relativistic cosmological problem follow the recommendations of [13]. The source code of `flrw_path2_verification.py` is distributed as part of the corpus.

CONFLICT OF INTEREST

The author declares no conflict of interest.

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Note on ordering. The bibliography is ordered in three conceptual blocks [L-35-ext]: (1) fundamental classical works (Bianchi, Noether, Friedmann, Lemaître, Lovelock, Wald, Weinberg, Planck 2018) — by year or close to year; (2) author's preprints in the ODTOE corpus (Pankratov A.S.) — in order of first citation in the text; (3) methodological and accompanying sources (Baumgarte–Shapiro, Jacobson).

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